Chromatic number using complete graph

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*ABSTRACT: A proper coloring of a graph requires the assignment of colors to the vertices of the graph in such a way that no two adjacent vertices are assigned the same color. This requirement leads to the construction of a complete graph whose vertices represent the color classes of the proper coloring. We call this graph as Color Class Complete Graph (CCCG). Finding the chromatic number of a graph requires the construction of the smallest CCC graph. In this paper, we find the chromatic number of the p-petal graph, when p = 3, using the CCC Graph. We also find the chromatic number of the line graph, middle graph and the total graph of the 3-petal graph.*

***KEY WORDS:*** *proper coloring, chromatic number, p-petal graph, line graph, middle graph, total graph*

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1. **INTRODUCTION**

Let be a simple graph with vertex set and edge set . A proper coloring of the vertices of is the assignment of colors from the set such that no two adjacent vertices are assigned the same color. The chromatic number of the graph is , the smallest possible integer.

A proper coloring of partitions its vertex set into a color class set of disjoint color classes. Brooks1 defined the chromatic number of , denoted as the minimum cardinality of , that is the smallest possible value of . For the smallest value of , it is required that at least one vertex in is adjacent to at least one vertex in for . This requirement leads to the construction of the smallest possible complete graph in which each vertex is represented by a color class, and we call it the color class complete graph (CCCG).

The chromatic number of the Petersen graph is 3. We can construct the three vertex color class complete graph with color classes

Refer to Figure 1.

**Figure 1: Petersen graph and its color class complete graph**

A petal graph is a connected graph  with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a -regular graph and the set of vertices of degree two induces an empty graph2. A petal graph of vertices and petals with petal sequence is said to be a *-*petal graph3,denoted , if every petal in is of size and with .In a -petal graph the petal size is always odd. In this paper, we consider *-*petal graphs with petals. Figure 2 shows two different versions of the *-*petal graph with the same number of petals. The *-*petal graphs are planar if the number of petals is even. **The** *-*petal graph with the number of petals **has** **vertices and** **edges.**

**Figure 2: Two versions of a *-*petal graph with 3 petals**

In the works of Whitney4 originated the concept of line graph. Consider an [undirected graph](https://en.wikipedia.org/wiki/Undirected_graph) where and . The line graph of  is the graph  in which each vertex in represents an edge in . Two vertices in are connected by an edge if their corresponding edges in are incident with a common vertex in .

Consider a -petal graph with number of petals . This graph has vertices and edges. The line graph of the -petal graph is clearly not another -petal graph, because it fails to satisfy the required conditions for a petal graph. The maximum degree of the vertices of the line graph is 4 and the minimum degree is 3. There are vertices and edges in . Figure 3 shows the line graph of the *-*petal graph with 3 petals. The -petal graph has 9 vertices and 12 edges, whereas its line graph has 12 vertices and 21 edges.

**Figure 3: The line graph of a -petal graph with 3 petals**

T. Hamada and I. Yoshimura5 introduced the middle graph of a graph , denoted as constructed in such a way that the vertex set of is the union of the vertex set and the edge set, that is . Two vertices of are adjacent if either

1. their corresponding edges are incident with the same vertex in or
2. their corresponding vertex and edge are incident in .

The middle graph of the -petal graph is clearly not another -petal graph. The maximum degree of the vertices of the middle graph is 6 and the minimum degree is 2. There are vertices, of which vertices are of degree six, vertices of degree five, 2 vertices of degree three and vertices of degree two. There are edges in M. Figure 4 shows the middle graph of the *-*petal graph with 3 petals. The core cycle of the middle graph has vertices of degree six and vertices of degree three.

**Figure 4: The middle graph of a *-*petal graph with 3 petals**

Mehdi Behzad6 defined the total graph of a graph. The total graph of a petal graph is one whose vertex set . Two vertices are adjacent in if either

1. their corresponding vertices in are adjacent in or
2. their corresponding edges are incident with the same vertex in or
3. their corresponding vertex and edge are incident in .

The total graph of the -petal graph is clearly not another -petal graph. The maximum degree of the vertices of the total graph is 6 and the minimum degree is 4. There are vertices, of which vertices are of degree six, vertices of degree five and vertices of degree four. There are edges in T. Figure 5 shows the total graph of the *-*petal graph with 3 petals. The vertices of degree six form the core cycle of the total graph.

**Figure 5: The total graph of a *-*petal graph with 3 petals**

1. **MAIN RESULTS**

In this paper, we consider the *-*petal graphs only. **V. Kolappan7 has already proved t**he first theorem using another method. We find these results using the complete graph in this work.

1. The chromatic number of a *-*petal graph is three.
2. The chromatic number of the line graph of a *-*petal graph three if is even and four if is odd.
3. The chromatic number of the middle graph of a *-*petal graph is four.
4. The chromatic number of the total graph of a *-*petal graph is four.

***Theorem 1***

*The chromatic number of the -petal graph is given by*

Let be a *-*petal graph with petals. There will be vertices in the vertex set of the *-*petal graph. The maximum degree of the vertices of a *-*petal graph is 3. The core of the *-*petal graph is an even cycle, irrespective of the number of petals being even or odd. Every petal starts from an odd sufixed vertex and ends at the even suffixed vertex. From the construction of the *-*petal graph, every odd suffixed vertex is adjacent to **two** even suffixed vert**ices** and a central vertex, every even suffixed vertex is adjacent to **two** odd suffixedverti**ces** and a central vertex, and every central vertex is adjacent to an odd and an even vertex. **The petals form the vertex group and the core cycle of the petal graph forms the vertex group .**

**Figure 6: The chromatic number of the *-*petal graph with 6 petals is 3**

We can use the complete graph with its three vertices represented by the three color classes , and . Refer Figure 6. Let the color class contains each of the odd suffixed vertices from , the color class contains each of the even suffixed vertices from and the color class contains each of the central vertices from that is adjacent to an even vertex and an odd vertex.








This color class complete graph represents the proper coloring of the *-*petal graph and fixes its chromatic number to be three.

***Theorem 2***

*The chromatic number of the line graph of the -petal graph is given by*

Consider the *-*petal graph with petals. The line graph of the *-*petal graph has vertices , , , …, and of degree four that form the core of the graph, and vertices , , , …, and of degree three that lie on the exterior to the core of the graph as depicted in Figures 7 and 8. We have , where and . The vertices , and form a complete graph as an induced subgraph. This forces the minimum number of colors required for a proper coloring of to be three.

***Case 1:*** *The number of petals is even*

Consider the complete graph with color classes , and . The vertices of are included in the color classes as follows:

1. , where

Refer Figure 7.

**Figure 7: The chromatic number of the line graph**  **with even petals is three**

Thus, the chromatic number of the line graph of the *-*petal graph with even number of petals is three.

***Case 2:*** *The number of petals is odd*

Consider the complete graph with color classes , and . The vertices of are included in the color classes as follows:

1. , where

The vertex is adjacent to , and , each of which belongs to the three color classes , and , so needs the fourth color class , thus extends into . Hence, the fourth color class is given by

Refer Figure 8.

**Figure 8: The chromatic number of the line graph**  **with even petals is four**

Thus, the chromatic number of the line graph of the *-*petal graph with odd number of petals is four.

***Theorem 3***

*The chromatic number of the middle graph of the -petal graph is given by*

Consider the *-*petal graph with petals. The middle graph of the *-*petal graph has vertices , , , …, and of degree six; vertices , , , …, and of degree five; vertices , , , …, and of degree three and vertices , , …, and of degree two. The vertices of degree six form the core cycle of the middle graph.

Consider the central vertex of minimum degree. It forms a together with and . So, the chromatic number is at least three. The neighbouring vertices and of are of degree five.

**Figure 9: The chromatic number of the middle graph is four**

Consider the vertex . This vertex forms a together with , and . So, the chromatic number is at least four. The neighbouring vertices and of are of degree six. Consider the vertex . This vertex fails to form a together with its neighbouring vertices , , , and .

This arguement is true for all central vertices , . This forces the minimum number of colors required for a proper coloring of to be four. This prevents the possibility of the chromatic number being five. Hence, the chromatic number is four.

Consider the complete graph with color classes , , and . The vertices of are included in the color classes as follows:

Refer Figure 9.

Thus, the chromatic number of the middle graph of the *-*petal graph is four.

***Theorem 4***

*The chromatic number of the total graph of the -petal graph is given by*

Consider the *-*petal graph with petals. The total graph of the *-*petal graph has vertices , , , …, , , , , , …, and of degree six; vertices , , , …, and of degree five; and vertices , , …, and of degree four. The vertices of degree six form the core cycle of the total graph.

**Figure 10: The chromatic number of the total graph is four**

Consider the central vertex of minimum degree. It forms a together with and . So, the chromatic number is at least three. The neighbouring vertices and of are of degree five. Consider the vertex . This vertex forms a together with , and . So, the chromatic number is at least four. The neighbouring vertices , , , and of are of degree six. Consider the vertex . This vertex fails to form a together with its neighbouring vertices , , , , and .

This arguement is true for all central vertices , . This forces the minimum number of colors required for a proper coloring of to be four. This prevents the possibility of the chromatic number being five. Hence, the chromatic number is four.

Consider the complete graph with color classes , , and . The vertices of are included in the color classes as follows:

Refer Figure 10.

Thus, the chromatic number of the total graph of the *-*petal graph is four.

# CONCLUSIONS AND RECOMMENDATIONS

The *-*petal graph is particularly interesting because of its planarity properties. When the *-*petal graph is not planar. Its also interesting because of its feasibility in getting the results. We have still lot more variants of the *-*petal graph and they are waiting for interesting results.

We hope that the following recommendations give important results:

1. Finding the b-chromatic number of the *-*petal graph, its line graph, middle graph and total graph.
2. Finding the chromatic and the b-chromatic number of graph such as the complementary graph, split graph and star graph that can be constructed from the *-*petal graph.
3. Finding the lower bounds for the chromatic number and the upper bounds b-chromatic number for the *-*petal graphs, when .
4. There are many variants for the chromatic number. We can find these chromatic numbers for all the above graphs

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