OPEN ACCESS

International
Journal
Of Modern Engineering Research (IJMER)

Application of Graph Theory to Formation Control Based on Relative Position Information

Dao Thi My Linh, Tran Minh Hai*, Ngo Thi Anh Hang
Faculty of Engineering and Technology, Thai Binh University,
Hung Yen province, Viet Nam

ABSTRACT
Graph theory serves as a fundamental framework for the design and analysis of formation control systems, particularly in multi-agent networks. By modeling interactions using graph structures, the inter-agent relationships are represented in a clear and systematic manner, providing comprehensive insights into the mechanisms of connectivity and information exchange within the system. This enables the development of control strategies tailored to specific formation objectives. This study focuses on three key aspects: (1) fundamental concepts of graph theory, including undirected and directed graphs, incidence matrices, adjacency matrices, degree matrices, and Laplacian matrices; (2) an overview of formation control methods based on relative position information; and (3) the formulation and validation of formation control laws using relative position data in ddd-dimensional space. Simulation results in MATLAB demonstrate that the proposed approach achieves stable formation maintenance and fast convergence, even when agents are initialized at random positions. These findings highlight the potential for broad application in autonomous robotic systems, unmanned aerial vehicles (UAVs), and coordinated mobile platforms operating in real-world environments.
Keywords:*Graph Theory; Multi-Agent Systems; Formation Control;*

Date of Submission: 15-09-2025 Date of acceptance: 30-09-2025

I. Introduction

In recent years, formation control for Multi-Agent Systems (MAS), including mobile robot teams, unmanned aerial vehicles (UAVs), and autonomous ground vehicles, has emerged as a research direction receiving significant attention from the scientific and engineering communities. The fundamental framework for addressing this problem lies in graph theory, which provides powerful mathematical tools for modeling and analyzing the interaction topology among agents.

In graph-based modeling, each agent is represented as a vertex (node), while communication or information-sharing links between agents are described as edges. Graph-theoretic structures such as incidence matrices and Laplacian matrices play a critical role in characterizing system dynamics and assessing stability properties. By leveraging graph-theoretic representations, inter-agent interactions can be described in a clear and structured manner, offering a comprehensive view of the connectivity and information-exchange mechanisms within the networked system.

Formation control is defined as a cooperative control strategy that enables one or multiple groups of agents to perform common tasks while maintaining a prescribed geometric configuration in space [1]. Originating from applications in intelligent highway systems, formation control has been investigated to allow autonomous vehicles to self-organize into platoons, maintain fixed inter-vehicle distances, and synchronize velocities [2]. Once the formation is established, high-speed motion can be achieved safely—preventing collisions between vehicles—and minimizing speed fluctuations, thereby improving fuel efficiency.

Formation control has been widely applied across various domains, including security patrol, environmental monitoring, search and rescue in hazardous environments, as well as military and transportation industries. The recent development of UAV technology has further introduced emerging applications such as geological exploration, disaster search and rescue, and area surveillance. These tasks can be performed more effectively and reliably when UAVs operate in predefined formations [3]. In military applications, groups of autonomous vehicles are required to maintain specific formations for area coverage and reconnaissance; in satellite cluster control, formations reduce propulsion fuel consumption while enhancing sensing capability and overall operational efficiency of the system [4]; in automated highway systems, traffic throughput can be significantly improved if vehicles are able to form platoons traveling at a desired velocity while maintaining fixed inter-vehicle spacing [5].

In formation control, two main control architectures are commonly employed: centralized control and distributed control [10]. For large-scale and complex Multi-Agent Systems (MAS), centralized control is often difficult to implement or even infeasible. Consequently, distributed formation control has gained broader attention due to its self-organizing capability, ease of implementation, and high reliability. The primary objective of formation control is to generate suitable control commands to guide multiple agents so that the required constraints on their states are satisfied [6].

From a mathematical perspective, formation control for MAS is developed based on concepts from graph theory and consensus dynamics [7]. Graph theory serves as an effective tool for describing the spatial configuration of MAS formations as well as the sensing, communication, and control topology among agents in a distributed structure. To achieve coordinated behavior, each agent must exchange information with its neighboring agents to reach an agreement on certain shared objectives. In formation control problems, the control variables may be the absolute positions [8],[9], the relative positions, or the inter-agent distances [6],[10].

To address the formation control problem, numerous approaches have been proposed; however, this work develops a distributed formation control law using absolute position information as the control input, combined with graph-based modeling to define the inter-agent interaction topology. Fundamental concepts of graph theory are presented and synthesized based on [11],[12]. In applying graph theory to formation control, both directed and undirected graphs can be employed to represent information exchange among agents: directed graphs are used for unidirectional interactions $\vec{G} = \left(V, \vec{E}\right)$, whereas undirected graphs are used for bidirectional interactions $G = \left(V, E\right)$ [11].

2.1 Theoretical Preliminaries.

2.1.1 Undirected Graph

Consider a multi-agent system formation represented by a simple undirected graph comprising a set of vertices and a set of edges.

Consider a multi-agent system formation represented by a simple undirected graph *G = (V, E)* comprising a set of vertices $V = \{v_1, v_2, ..., v_n\}$ with $|V| = n > 0$ elements, a set of edges $E = \{(v_i, v_j) | i, j = 1, ..., n, i \neq j\} \in V \times V$ with $|E| = m$ elements.

Let $v_i \in V$ và $\left(v_i, v_j\right) \in E$ denote a vertex and an edge of graph *G*. A graph is undirected if $\left(v_i, v_j\right) \in E$ implies $(v_j, v_i) \in E$ .When considering multiple graphs, the vertex set and edge set of G are denoted by *V(G)* and *E(G)* respectively. In certain contexts, for simplicity, vertex *i*, and edge (v_i, v_j), may be represented as *(i,j)* or e_{ij}. Each graph *G* admits a corresponding geometric representation, where small circles denote the vertices $v_i \in V$ and line segments (or arcs) connect v_i and v_j whenever $(v_i, v_j)) \in E$. if $(v_i, v_j) \in E(G)$ the vertices v_i, v_j are said to be adjacent $(v_i \sim v_j)$, *vertex viis said to be incident to edge* (v_i, v_j). Two distinct edges that share a common vertex are called adjacent edges.

In a simple undirected graph, there exists at most one edge between any two vertices. In contrast, an undirected multigraph, defined as *G = (V, E)*, permits multiple edges between the same pair of vertices

A weighted graph *G* is formally defined as an ordered triple *(V,E,A)* where, in addition to the vertex set V and the edge set E, there exists a weight set $\{\omega_{ij} \in \mathbb{R}_+ | i \neq j, i, j \in V\}$. Each edge $(i, j) \in E$ is assigned a positive weigh $\omega_{ij} > 0$ while non-adjacent vertex pairs are assigned a weight of zero $\omega_{ij} = 0$

The relationships between the vertices and edges of the graph are typically described using matrix representations, which play a critical role in characterizing the connectivity structure of formation graphs[11]. In particular, the adjacency matrix $A(G) = [a_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$ directly encodes the interconnections among the graph vertices.

$$a_{ij} = \begin{cases} \omega_{ij} & if \ \left(v_i, v_j \in E\right) \\ 0 & otherwise \end{cases} \tag{1}$$

For an undirected graph.

$$a_{ij} = a_{ji} = \begin{cases} 1 & if\left(v_i, v_j \in E\right) \\ 0 & otherwise \end{cases} \quad (2)$$

The degree of a vertex in a graph is defined as the number of edges incident to that vertex, denoted as *deg(v)*
In an undirected graph, the degree matrix *D(G)* is a diagonal matrix representing the degree of each vertex. For a graph with n vertices*, D(G)* is an $n\times n$ square matrix. [11]

$$D[i,j] = \begin{cases} \deg(v_i) & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} \quad (3)$$

Here, $deg(v_i)$denotes the degree of vertex*i* (he number of edges incident to that vertex) and*D* is a diagonal matrix with nonzero entries only along its main diagonal

2.1.2 Directed Graph.
A directed graph *G* is defined as an ordered pair*(V,E),* where *V=V(G)* denotes the set of vertices and $E = E(G) \in V\times V$denotes the set of directed edges. A directed edge$(u,v) \in E(u\neq v)$is represented geometrically by an arrow from vertex *u* tto vertex *v*. In directed graphs*, (u,v)*and*(v,u)*are distinct edges that may coexist, and the presence of$(u,v) \in E$does not imply$(v, u) \in E$. [11]
Most definitions for undirected graphs naturally extend to directed graphs. For each vertex $u \in V$, the out-degree is defined as$deg^+(u) = |N^+(u)|$, where$N^+(u) = \{v|\ (u,v) \in E\}$denotes the out-neighbor set of *u*. Similarly, the in-degree $deg^-(u)$and in-neighbor set $N^-(u)$are defined analogously. A directed graph *G* is called balanced if$deg^+(u) = deg^-(u)$ for all$u \in V$ [11]
The in-degree matrix of a directed graph is defined as

$$D(G) = diag(deg^-(v_1),\ldots,deg^-(v_n)) \quad (4)$$

The adjacency matrix of a directed graph is determined analogously to that of an undirected graph, but
$a_{ij} \neq a_{ji}$

2.1.3 Laplacian matrix
In graph theory, the Laplacian matrix is a fundamental algebraic tool that mathematically characterizes the connectivity structure of a graph and plays a critical role in stability analysis, consensus studies, and formation control. The general formulation of the Laplacian matrix, applicable to various classes of graphs, is commonly expressed as follows [11]

$$L \,\square\, D - A \quad (5)$$

In here,$L = [\ell_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$represents the Laplacian matrix of the graph *G*

2.1.4 Incidence Matrix
A graph *G = (V, E, A)*with $|V| = n$vertices and $|E| = m$ edges. The incidence matrix$\mathbf{H}=[h_{ki}]^{m\times n} \in R^{m\times n}$represents the vertex–edge relationships of the graph, where each row corresponds to an *edge* $(e_1,e_2,\ldots,e_m)$ and each column corresponds to a vertex [11]

$$h_{ij}\begin{cases} 1, & if \quad e_k = \left(v_i, v_j\right) \\ -1, & if \quad e_k = \left(v_j, v_i\right) \\ 0, & otherwise \end{cases} \quad (6)$$

2.1.5Overview of Formation Control Based on Relative Position
Necessary assumptions for multi-agent systems in relative-position-based formation control:
(1) Measurement: Each agent is equipped with a local reference frame aligned with the global coordinate frame; the agents need not know the origin of the global coordinate frame. Within their local frames, each agent can measure relative position (position error vector) and relative velocity vectors of a subset of neighboring agents. Because all local frames share the same orientation as the global frame, the measured relative position vectors are invariant across frames. (2) Interaction topology: The sensing relationships among agents are modeled by a connected undirected graph *G = (V,E).* The desired formation is specified by a set of prescribed relative position vectors assigned to the edges of *G*.

In the case where each agent in the formation is modeled as a first-order integrator

$$\dot{p}_i = u_i;\ i = 1,\ldots,n, \qquad (7)$$

In here, $p_i \in R^d$và $u_i \in R^d$denote the position and the control input of agent*i*expressed in the global reference frame gΣ.

From the aforementioned sensing assumptions, each agent *i* can measure

$$z_{ij} = p_j - p_i, \forall j \in N_i.$$

The desired formation is specified by the set $\Gamma = \{z_{ij}^*\}(i,j) \in E,$ the target for each agent is to achieve a formation configuration in which all desired relative position vectors are satisfied. $z_{ij} = p_j - p_i = z_{ij}^*, (i,j) \in E$

Set $\Gamma = \{z_{ij}^*\}(i,j) \in E,$ is said to be feasible if ε_{p*} is nonempty

$$\varepsilon_{p*} \ \square\ \left\{ p \in R^{dn} \mid p_j - p_i = z_{ij}^*\ ij, \forall (i,j) \in E \right\}$$

The set Γ is assumed to be feasible, and we define $p^* = vec\left(p_1^*,\ldots,p_n^*\right)$as an element of ε_{p*}

Assume that each agent only knows a set of predefined position errors $z_{ij}^* = p_j^* - p_i^*, \forall j \in N_i$ without knowing $p_j^*\ \&\ p_i^*$

The problem is to drive *p*to a formation that differs from*p**by a translation, or equivalently, to ensure that*p(t)*converges to the set ε_{p*}khi $t \to +\infty$.For this problem, the formation control law is designed as follows.

$$\begin{aligned} u_i &= k_p \sum_{j \in N_i} a_{ij}\left(z_{ij} - z_{ij}^*\right) \\ &= k_p \sum_{j \in N_i} a_{ij}\left(\mathrm{p}_j - p_i\right) - \left(\mathrm{p}_j^* - p_i^*\right) \end{aligned} \qquad (8)$$

With $k_p > 0$, $\delta = p^* - p$

$$\dot{\delta} = -k_p\left(L \otimes I_d\right)\delta \qquad (9)$$

According to consensus theory, if *G* is a connected graph, then $\delta(t) \to \delta^* = 1n \otimes \bar{\delta}$

In there $\bar{\delta} = \frac{1}{n}\sum_{i=1}^{n}\left(p_i^*(0) - p_i(0)\right)$is a constant vector. As a result, $p^* - p(t) \to \delta^*,\ t \to +\infty$

Hay $p(t) \to p^* - \delta^*,\ t \to +\infty$, That is,*p(t)* converges to a fixed configuration belonging to the set ε_{p*}

In the case where each agent in the formation has second-order integrator dynamics

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= u_i, i = 1,\ldots,n. \end{aligned} \qquad (10)$$

p_i, v_ivà $u_i \in R^d$lần lượt là vị trí, vận tốc và tín hiệu điều khiển của tác tử i viết trên hệ qui chiếu toàn cục gΣ. Khi đó luật điều khiển viết cho từng tác tử như sau:

p_i, v_ivà $u_i \in R^d$denote the position, velocity, and control input of agent i, respectively, expressed in the global reference framegΣ. Then, the control law for each agent can be written as follows

$$u_i = -k_1\left(p_i - p_j - \left(p_i^* - p_j^*\right)\right) - k_2 v_i,\ i = 1,\ldots,n,$$

k_1, $k_2$are positive real constants

By further performing the variable transformation $\delta = p_i - p_i^*$ the following equation is obtained

$$\begin{aligned} \dot{\delta} i &= v_i \\ \dot{v}_i &= -k_1 \sum_{j \in N_i}\left(\delta_i - \delta_j\right) - k_2 v_i,\ i = 1,\ldots,n \end{aligned} \qquad (11)$$

The above control law drives the agents toward consensus $\delta_i \to \delta^*, v_i \to 0_d$, this is equivalent to*p(t)* converges to a fixed configuration $p_i(t) \to p_i^* + \delta^*, t \to +\infty$.

III. Simulation Results

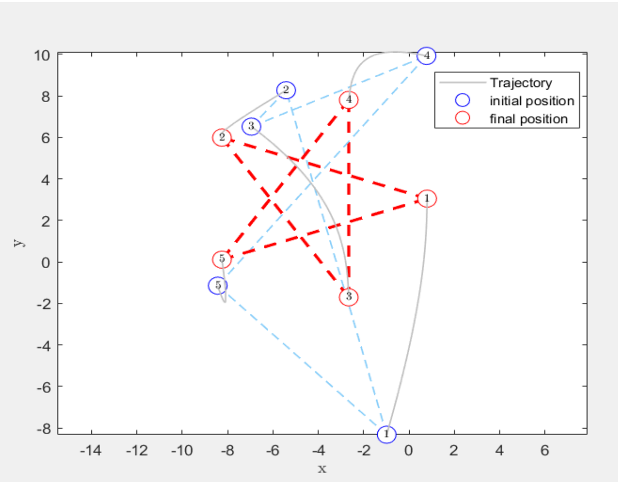
Simulations were carried out in MATLAB to verify the formation control law based on relative positions in 2D and 3D spaces. The system consists of five agents, each moving directly toward its assigned position in space. The target positions are arranged at the vertices of a 5 -pointed star inscribed in a regular pentagon. The initial positions are randomly generated. Each agent in the formation is modeled as a first-order integrator and is governed by the control laws (8) and (9). The adjacency matrix H, of size 5×5, describes the relationships between the vertices and the edges in the graph. In this case, H is configured as a directed circular graph, where each agent receives control information from its preceding neighbor, ensuring strong connectivity and closure of the multi-agent system.

The matrix H is presented as follows

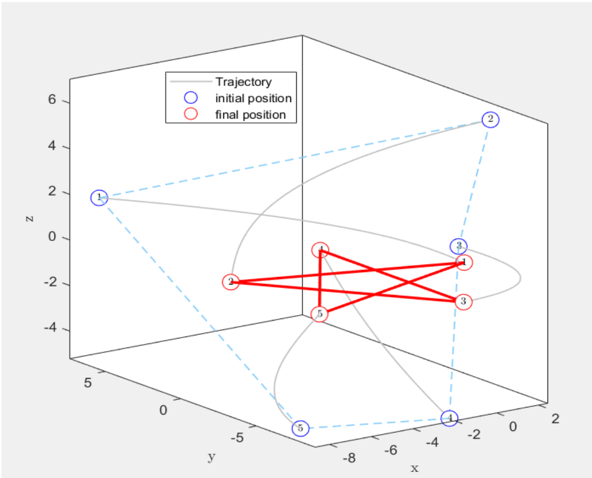
```
H =

    -1     1     0     0     0
     0    -1     1     0     0
     0     0    -1     1     0
     0     0     0    -1     1
     1     0     0     0    -1
```

Processing the matrix H in MATLAB using a for loop

```
n=5
H = -eye(n)
for i=1:1:n-1
H(i,i+1) = 1
end
H(n,1) = 1
```



(a) Formation in 2D space



(b) Formation in 3D space

Figure 1: Simulation of the formation control algorithm based on relative positions

The agents successfully converged to their respective target points without significant deviations or oscillations. This demonstrates the correctness and effectiveness of the applied control law

IV. Conclusion

This study clarifies the central role of graph theory in the design and analysis of distributed formation control strategies for multi-agent systems. Modeling the interconnection structure using the adjacency matrix not only provides a rigorous mathematical foundation but also enables a clear characterization of agent interactions within the system. The designed closed directed graph structure ensures strong connectivity, allowing each agent to update its state based on position information received from its preceding neighbor in the cycle.

The formation control law (9) is formulated using relative position information between agents, with the objective of driving the entire system toward a desired geometric configuration—specifically, converging the agents to the vertices of a regular polygon inscribed in a circle of radius 5. Simulation results performed in MATLAB for both 2D and 3D spaces with five mobile agents demonstrate the effectiveness of the proposed approach. Despite random initialization in space, the agents rapidly converge to the desired formation with negligible error. These results not only validate the theoretical soundness of the algorithm but also highlight its potential for practical applications in cooperative robotic systems, UAVs, and autonomous vehicles operating in unstructured environments.

Overall, this work strengthens the link between graph theory and modern formation control, laying a foundation for future extensions involving adaptive or fully distributed control strategies, as well as integration with dynamic environmental conditions and real-world disturbances

References

[1]. X. Wang, Z. Zeng, and Y. Cong, Multi-agent distributed coordination control: Developments and directions via graph viewpoint, Neurocomputing, vol. 199, pp. 204-218, 2016.

[2]. J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. IEEE Transactions on Automatic Control, 49(9):1465–1476, 2004.

[3]. B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx. Rigid graph control architectures for autonomous formations. Control Systems Magazine, 28(6):48–63, 2008

[4]. Martin M, Kilberg S, Winter J, editors. TechSat 21 and Revolution izing Space Missions Using Microsatellites. American Institute of Aeronautics and Astronautics; 2001.

[5]. Bender JG. An overview of systems studies of automated highway systems. IEEE Transactions on Vehicular Technology. 1991;40:82– 99.

[6]. K.-K. Oh, M.-C. Park, and H.-S. Ahn, A survey of multi-agent formation control,Automatica, vol. 53, pp. 424-440, 2015.

[7]. H. S. Ahn, Formation control. Springer International Publishing, 2020.

[8]. Liu Y, Bucknal R. A Survey of Formation Control and Motion Planning of Multiple Unmanned Vehicles. Cambridge University. 2013.

[9]. Xian Y, Na X, Sun Z, Chen J. Formation control and collision avoidance for multi-agent systems based on position estimation. Beijing Institute of Technology.2015

[10]. Hoa N T T, Minh N T. Distributed formation control of multi-agent systems based on rigidity graph theory applied in moving target tracking. TNU - University of Technology. 2023.

[11]. Trinh HM, Nguyen MH. Điều khiển hệ đa tác tử 2023.

[12]. Godsil C, Royle G. Algebraic Graph Theory. Springer-Verlag New York, Inc: University of Waterloo Waterloo, Ontario N2L 3Gl Canada University of Western Australia Nedlands, Western Australia 6907 Australia 2001.