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A solution of one-dimensional dispersion phenomenon by Homotopy Analysis Method

Kajal Patel[1], M. N. Mehta[2], Twinkle R. Singh[3]

[1](Research scholar, Department of Applied Mathematics and Humanities, S. V. National Institute of Technology Surat-395007, Gujarat, India)

[2](Department of Applied Mathematics and Humanities, S. V. National Institute of Technology Surat-395007, ,Gujarat, India)

ABSTRACT: *The present paper discusses solution of dispersion phenomenon by using Homotopy analysis method. Solution represents concentrations of any contaminated or salt water disperse in homogenous porous media saturated with fresh water. The solution of non-linear partial differential equation has been obtained in term of series solution of exponential function of X and time T under assumption of guess value of concentration of contaminated or salt water. Here solution converges for parameter* $\varepsilon = 1$ *for embedded parameter* $\hbar = 0.1$. *The graphical presentation is given by using Maple coding. It is concluded that concentration of contaminated or salt water dispersion is decreasing when distance X as well as time T increasing and convergence of the solution has been discussed.*

Keywords: *Advection, Diffusion, Homotopy analysis Method*

I. INTRODUCTION

The present paper discusses the solution of longitudinal dispersion phenomenon, which arising in the miscible fluid flow through homogenous porous media. The problem of solute dispersion during underground water movement has attracted interest from the early days of this century [1], but it was only since 1905 in general topic of hydrodynamic dispersion or miscible displacement becomes one of the more systematic studies. The phenomenon of the dispersion has been receiving good attention from hydrologist, agriculture, environmental, mathematicians, chemical engineering and soil scientists. The specific problem of fluid mixing in fixed bed reactors has been investigated by Bernard and Wilhelm [2]. Kovo [3] has worked with the parameter to be modeled in the longitudinal or axial dispersion coefficient D in chemical reactors model.

The fundamental interest of this paper is to find concentration of contaminated or salt water. The term concentration expresses a measure of the amount of a substance within a mixture. The dispersion process is associated with molecular diffusion and mechanical dispersion. Molecular diffusion is the spreading caused by the random molecular motion and collisions of the particles themselves and mechanical dispersion is the spreading of a dissolved component in the water phase by variations in the water velocity (i.e. flow of a fluid). These two basic mechanisms molecular diffusion and mechanical dispersion cause a concentration front of fluid particles to spread as it advances through the porous media. These two combine processes of molecular diffusion and mechanical dispersion are known as hydrodynamic dispersion or dispersion.

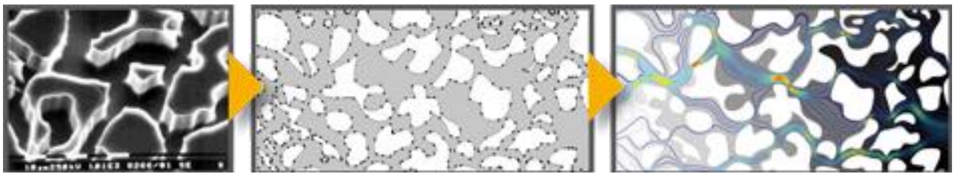

Fig 1: The geometry of microscopic pores, where velocity distributions in different pore size.

When groundwater flows, the actual microscopic velocity in the pores varies widely in space even when the Darcy macroscopic velocity is constant. The result is more intense mixing, which is called hydrodynamic dispersion. Figure 1 gives a schematic view of the trace movement on macroscopic level. This phenomenon can be observed in coastal areas, where the fresh waterbeds are gradually displaced by seawater. This phenomenon plays an important role in the seawater intrusion into reservoir at river mouths and in the underground recharge of groundwater.

Most of the works reveal common assumption of homogenous porous media with constant porosity, steady seepage flow velocity and constant dispersion coefficient. For such assumption Ebach and White [4] studied the longitudinal dispersion problem for an input concentration that varies periodically with time. Hunt [5] applied the perturbation method to longitudinal and lateral dispersion in no uniform seepage flow through heterogeneous aquifers. Mehta and Patel [6] applied Hope-Cole transformation to unsteady flow against dispersion of miscible fluid flow through porous media. Marino [7] considered the input concentration varying exponentially with time. Eneman et al.[8] provided analysis for the systems where fresh water is overlain by water with a higher density in coastal delta areas. Experimental evidence of such lenses was given by Lebbe et al. [9], and Vandenbohede et al. [10]. Meher and Mehta [11, 12] studied the Dispersion of Miscible fluid in semi infinite porous media with unsteady velocity distribution using Adomain decomposing method.

The present paper discusses the approximate analytical solution of the nonlinear differential equation for longitudinal dispersion phenomenon which takes places when miscible fluids (contaminated or salt water) mix in the direction of flow. The mathematical formulation of the problem yields a non linear partial differential equation. The analytical solution has been obtained by using homotopy analysis method.

II. Statement of the Problem

Considering dispersion of contaminated or salt water with concentration $C(x,t)$ flowing in x-direction, dispersion taking place in porous media saturated with fresh water. Hence it will be miscible fluid flow through homogenous porous media. Therefore, it will obey the Darcy's law, which dates back to 1856 [13]. The following assumptions have been made for present analysis (Schidegger 1954, Day 1956, deJony 1958) [14,15, 16]:

- The medium is homogenous.
- There is no mass transfer happen between the solid and liquid phases.
- The solute transport across any fixed plane, due to microscopic velocity variation in the flow tube, may be quantitatively expressed as the product of a dispersion coefficient and the concentration gradient.

To find concentration of the dispersing contaminated or salt water as a function of time t and distance x, as the two miscible fluids flow through homogenous porous media. Since the mixing (contaminated or salt water and fresh water) takes place both longitudinally and transversely. Dispersion adds a spreading effect to the diffusion effects. Science dispersion is driven by the mean flow of the water, the dispersion coefficients related to the characteristic length or pore length L. In three dimensions, the spreading caused by dispersion is greater in the direction of the flow than in the transverse direction. One dimensional treatment of these systems avoids treatment of a radial or transverse component of dispersion. We only consider the dispersion phenomenon in the direction of flow (i.e. longitudinal dispersion), which takes places when miscible fluids flow in homogenous porous media.

III. Mathematical Structure

According to Darcy's law, the equation of continuity for the mixture, in the case of incompressible fluids is given by Bear [1].

$$\frac{\partial \rho}{\partial t}+\nabla.\left(\rho\vec{V}\right)=0 \tag{1}$$

Where, ρ is the density for mixture [ML^{-3}], t is time [T] and $\vec{V}$ is the pore seepage velocity vector [LT^{-1}].

The equation of diffusion for a fluid flow through a homogeneous porous medium, without increasing or decreasing the dispersion of contaminated or salt water is given by

$$\frac{\partial C}{\partial t}=-\nabla.\left(C\vec{V}\right)+\nabla\cdot\left[\rho\bar{D}\cdot\left(\frac{C}{\rho}\right)\right] \tag{2}$$

Where C is the concentration of dispersing contaminated or salt water per unit volume [ML^{-3}] and $\bar{D}$ is the tensor coefficient of dispersion with nine components D_{ij} [L^2T^{-1}].

Since contaminated or salt water is flowing through a homogeneous porous medium at constant temperature, ρ may be considered as constant. Then

$$\nabla\cdot\vec{V}=0 \tag{3}$$

where $\vec{V}$ is velocity of contaminated /salt water dispersion.
Hence Eq. (2) may be written as

$$\frac{\partial C}{\partial t}=-\vec{V}\cdot\nabla C+\nabla\cdot\left[\bar{D}\nabla C\right] \tag{4}$$

When the seepage velocity $\vec{V}$ of contaminated or salt water is dispersing along the x-axis, then the non-zero components will be $D_{11}=D_L=\frac{L}{C_0^2}$, (coefficient of longitudinal dispersion and L is length of dispersion in flow direction) and $D_{22}=D_T$ (coefficient of transverse dispersion) and other D_{ij} are zero. From this assumption the equation (4) becomes,

$$\frac{\partial C}{\partial t}=-u\frac{\partial C}{\partial x}+D_L\frac{\partial^2 C}{\partial x^2} \tag{5}$$

where u is the component of seepage velocity of contaminated or salt water in x direction which is function of x and t and $D_L>0$. It has been observed that seepage velocity u is related with concentration of contaminated or salt water dispersion. We assume that seepage velocity u is directly proportional to $C(x,t)$ [11]

$$u=\frac{C(x,t)}{C_0} \tag{6}$$

where $1/C_0$ is constant of proportionality and the guess approximation of the concentration of contaminated or salt water dispersion. To get dimensionless form of equation (5) using the dimensionless variables

$$X=\frac{C_0 x}{L}, and\, T=\frac{1}{L}t$$

then equation (5) can be written as,

$$\frac{\partial C}{\partial T} = -C\frac{\partial C}{\partial X} + \varepsilon\frac{\partial^2 C}{\partial X^2}, \text{ where } \varepsilon = \frac{D_L C_0^2}{L} \text{ and } \varepsilon \in [0,1], 0 \le X \le 1, 0 \le T \le 1 \tag{7}$$

Since concentration C is decreasing as distance X increase for T > 0. It appropriate to choose guess value of concentration solution as, [12]

$$\square(X,T;\varepsilon) = \left(1 - \frac{1}{2}XT\right)e^{-X} + \varepsilon^m \tag{8}$$

Hence, the equation (7) together with boundary condition (8) represents the governing non-linear partial differential equation for concentration of the longitudinal dispersing material of miscible fluids flowing through a homogenous porous medium.

IV. THE SOLUTION WITH HOMOTOPY ANALYSIS METHOD

For one dimensional non-linear partial differential equation for longitudinal dispersion phenomenon, we assumed that the concentration $C(X,T)$ of the dispersing contaminated or salt water, at time T=0 is expressed as,

$$\square(X,T,\varepsilon = 0) = \left(1 - \frac{1}{2}TX\right)e^{-X} + \varepsilon^m, \text{ where } \varepsilon = 0 \text{ for the concentration of contaminated or salt water} \tag{9}$$

Now we apply the homotopy analysis method into the longitudinal dispersion phenomenon during miscible fluid flow through homogenous porous media. We consider the equation (7) as nonlinear partial differential equation as

$$\square\left[\square(X,T;\varepsilon)\right] = 0 \tag{10}$$

Where $\square$ is a non-linear operator, $\square(X,T;\varepsilon)$ is considered as unknown function which represent the concentration C of the dispersing contaminated or salt water at any distance X for given time T > 0, for $0 \le \varepsilon \le 1$. We use auxiliary linear operator $\Im\left[\square(X,T;\varepsilon)\right] = \frac{\partial \square(X,T;\varepsilon)}{\partial T}$ and initial approximation of concentration of dispersing contaminated or salt water $C_0(X,T) = \left(1 - \frac{1}{2}TX\right)e^{-X}$ to construct the corresponding zeroth order deformation equation. As the auxiliary linear operator $\Im$ which satisfies $\Im[C_4] = 0$, where C_4 is arbitrary constant. This provides a fundamental rule to direct the choice of the auxiliary function $H(X,T) \ne 0$, the initial approximation $C_0(X,T)$, and the auxiliary linear operator $\Im$, called the rule of solution expression. Establish the zero-order deformation equation of longitudinal dispersion phenomenon as [20],

$$(1-\varepsilon)\Im\left[\square(X,T;\varepsilon) - C_0(X,T)\right] = \varepsilon\hbar H(X,T)\square\left[\square(X,T;\varepsilon)\right] \tag{11}$$

where $C_0(X,T)$ denote an initial guess value of concentration of dispersing contaminated or salt water of the exact solution $C(X,T)$ which is our purpose to find it. Since $\hbar \ne 0$ an auxiliary parameter and $H(X,T) \ne 0$ an auxiliary function such that $\varepsilon \in [0,1]$ an embedding parameter. The auxiliary parameter $\hbar$ is providing a simple way to ensure the convergence of series. Thus it renamed $\hbar$ as convergence control parameter [20]. Let $\Im$ an auxiliary linear operator with the property that,

$\Im\left[\square(X,T;\varepsilon)\right] = 0$ when $C(X,T;\varepsilon) = 0$

When $\varepsilon = 0$, the zero-order deformation equation (11) becomes

$$\Im\left[\square(X,T;\varepsilon) - C_0(X,T)\right] = 0 \tag{12}$$

Which gives the first rule of solution expression and according to the initial guess $C_0(X,T) = \left(1 - \frac{1}{2}TX\right)e^{-X}$, it is straightforward to choose

$$\square(X,T;0) = C_0(X,T) \tag{13}$$

When $\varepsilon = 1$, since $\hbar \ne 0$, $H(X,T) \ne 0$ the zero-order deformation equation (7) is equivalent to

$$\square\left[\square(X,T;\varepsilon)\right] = 0 \tag{14}$$

which is exactly the same as the original equation (10) provided

$$\square(X,T;1) = C(X,T) \tag{15}$$

According to (13) and (15) as the embedding parameter ε increases from 0 to 1, solution $\square(X,T;\varepsilon)$ varies continuously from the initial guess value of the concentration $C_0(X,T)$ of dispersing contaminated or salt water to the solution $C(X,T)$ and its solution is assumed by expanding $\square(X,T;\varepsilon)$ in Taylor series with respect to ε as,

$$\square(X,T;\varepsilon) = \square(X,T;0) + \sum_{m=1}^{\infty} C_m(X,T)\varepsilon^m \tag{16}$$

Where, $$C_m(X,T) = \frac{1}{m!}\left.\frac{\partial^m \square(X,T;\varepsilon)}{\partial \varepsilon^m}\right|_{\varepsilon=0} \tag{17}$$

i.e. the concentration of dispersing contaminated or salt water is function of distance X and time T for any parametric value ε is expressed as, the concentration of dispersing contaminated or salt water at time $T=0$, $C_0(X,T)$ and sum of concentration of dispersing contaminated or salt water $C_1(X,T)$, $C_2(X,T)$,…at different time T for different value of parameter ε. Here, the series (16) is called homotopy-series; the series (16) is called homotopy series solution of $\square\left[\square(X,T;\varepsilon)\right]=0$ and $C_m(X,T)$ is called the mth-order derivative of $\square$. Auxiliary parameter $\hbar$ in homotopy-series (16) can be regard as iteration factor and is widely used in numerical computations. It is well known that the properly chosen iteration factor can ensure the convergence of homotopy series (16) is depending upon the value of $\hbar$, one can ensure that convergent of homotopy series, solution simply by means of choosing a proper value of $\hbar$ as shown by Liao [20]. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, the auxiliary function $H(X,T)$ are so properly chosen, the series (16) converges at $\varepsilon=1$.

Hence the concentration of dispersing water can be expressed as,

$$C(X,T)=C_0(X,T)+\sum_{m=1}^{\infty}C_m(X,T) \tag{18}$$

And $C_m(X,T)$ can be calculated by equation (23). This must be one of solution of original non-linear partial differential equation (7) of the concentration of dispersing contaminated or salt water problem in homogenous porous medium.

According to the definition (17), the governing equation can be deduced from the zero-order deformation equation (11), define the vector

$$\vec{C}_m=\{C_0(X,T),C_1(X,T),\ldots C_n(X,T)\}$$

Differentiating equation (11) m-times with respect to the embedding parameter ε and then setting $\varepsilon=0$ and finally dividing them by $m!$, we have the so-called m[th] order deformation equation of the concentration $C(X,T)$ will be as, [20]

$$\Im\left[C_m(X,T)-\chi_m C_{m-1}(X,T)\right]=\varepsilon\hbar H(X,T)R_m\left(\vec{C}_{m-1},X,T\right) \tag{19}$$

Where $$R_m\left(\vec{C}_{m-1},X,T\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\square\left[\square(X,T;\varepsilon)\right]}{\partial\varepsilon^{m-1}}\right|_{\varepsilon=0} \tag{20}$$

And $$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} \tag{21}$$

It should be emphasized that $C_m(X,T)$ for $m\ge 1$, is governed by the linear equation (20) with the linear boundary condition that came from original problem, which can solved by symbolic computation software Maple as bellow. The rule of solution expression as given by equation (8) and equation (11), the auxiliary function independent of ε can be chosen as $H(X,T)=1$, [20]

According to (15) and taking inverse of equation (19) the equation (20) becomes,

$$C_m(X,T)=\chi_m C_{m-1}(X,T)+\hbar\Im^{-1}\left[R_m\left(\vec{C}_{m-1},X,T\right)\right] \tag{22}$$

$$R_m\left(\vec{C}_{m-1},X,T\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N\left[\square(X,T;\varepsilon)\right]}{\partial\varepsilon^{m}}\right|_{\varepsilon=0} \tag{23}$$

In this way, we get $C_m(X,T)$ for m=1, 2, 3, … successively by using Maple software as,

$$C_1(X,T)=\frac{1}{12}hT\left(-T^2X^2+T^2X+3TXe^X-6Te^X+6TX-3T-6Xe^X-12-12e^X\right)e^{-2X} \tag{24}$$

$$C_2(X,T)=Th\begin{pmatrix}-0.5hXe^{2X}-0.25TXe^{2X}+0.052hT^3+0.146hT^3X^2e^X+0.58hT^2-0.92hT^2Xe^X+hTXe^X\\ +1.58hT^2e^X+2.5hTe^X-he^{2X}-0.5Te^{2X}-0.5Xe^{2X}-he^X-0.5hTe^{2X}+0.19hT^3e^X\\ -0.083T^2X^2e^X+0.5TXe^X+0.083T^2Xe^X-e^{2X}-0.083hT^2Xe^{2X}-0.25hT^2X^2e^X+1.5hT\\ -hT^2X+0.25hT^3X^2-0.32hT^3X+0.05hT^4X^2-0.017hT^4X-0.025hT^4X^3\\ -0.438hT^3Xe^X-e^X+0.5hTXe^{2X}-0.25Te^X+0.33hT^2e^{2X}\end{pmatrix}e^{-3X} \tag{25}$$

Using initial guess value of concentration from equation (8) and successive

$$C(X,T)=\left\{\begin{array}{l}\left(1-\frac{1}{2}XT\right)e^{-X}+\frac{1}{12}hT\begin{pmatrix}-T^2X^2+T^2X+3TXe^X-6Te^X\\ +6TX-3T-6Xe^X-12-12e^X\end{pmatrix}e^{-2X}\\ +Th\begin{pmatrix}-0.5hXe^{2X}-0.25TXe^{2X}+0.052hT^3+0.146hT^3X^2e^X+0.58hT^2\\ -0.92hT^2Xe^X+hTXe^X+1.58hT^2e^X+2.5hTe^X-he^{2X}-0.5Te^{2X}\\ -0.5Xe^{2X}-he^X-0.5hTe^{2X}+0.19hT^3e^X-0.083T^2X^2e^X+0.5TXe^X\\ +0.083T^2Xe^X-e^{2X}-0.083hT^2Xe^{2X}-0.25hT^2X^2e^X+1.5hT\\ -hT^2X+0.25hT^3X^2-0.32hT^3X+0.05hT^4X^2-0.017hT^4X\\ -0.025hT^4X^3-0.438hT^3Xe^X-e^X+0.5hTXe^{2X}-0.25Te^X+0.33hT^2e^{2X}\end{pmatrix}e^{-3X}+\ldots\end{array}\right\} \tag{26}$$

V. NUMERICAL AND GRAPHICAL SOLUTION

Numerical and graphical presentations of equation (26) have been obtained by using Maple coding. Fig 2 represents the graphs of concentration $C(X,T)$ vs. distance X, for T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 is fixed, and Table I indicates the numerical values. The fig 2 and the table 1, indicate the graphical representations of the longitudinal dispersion phenomenon of the concentration. The convergence of the homotopy series (16) is dependent upon the value of convergence-parameter $\hbar$ [17, 18, 19, 20]. Therefore we choose proper value of the convergence-parameter $\hbar = 0.1$ to obtain convergent homotopy-series solution [20].

Table I: Concentration of the Contaminated or Salt Water $C(X,T)$ For Different Distance X For Fixed Time T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0.

Distance X	$C(X,T)$ T= 0.1	$C(X,T)$ T= 0.2	$C(X,T)$ T= 0.3	$C(X,T)$ T= 0.4	$C(X,T)$ T= 0.5	$C(X,T)$ T= 0.6	$C(X,T)$ T= 0.7	$C(X,T)$ T= 0.8	$C(X,T)$ T= 0.9	$C(X,T)$ T=1.0
0.1	0.8674	0.8287	0.7887	0.7476	0.7054	0.6623	0.6183	0.5736	0.5282	0.4822
0.2	0.7814	0.743	0.7037	0.6635	0.6225	0.5807	0.5383	0.4953	0.4517	0.4077
0.3	0.7038	0.666	0.6274	0.5881	0.5482	0.5077	0.4667	0.4253	0.3834	0.3412
0.4	0.6338	0.5966	0.5589	0.5206	0.4818	0.4426	0.403	0.3631	0.3228	0.2822
0.5	0.5707	0.5343	0.4974	0.4602	0.4226	0.3847	0.3465	0.308	0.2692	0.2302
0.6	0.5137	0.4782	0.4424	0.4063	0.3699	0.3333	0.2964	0.2594	0.2221	0.1846
0.7	0.4624	0.4279	0.3932	0.3583	0.3231	0.2878	0.2523	0.2167	0.1809	0.1449
0.8	0.4162	0.3828	0.3492	0.3155	0.2816	0.2476	0.2135	0.1793	0.1449	0.1105
0.9	0.3745	0.3423	0.3099	0.2775	0.2449	0.2123	0.1795	0.1467	0.1138	0.0808
1.0	0.3369	0.3059	0.2748	0.2437	0.2124	0.1812	0.1498	0.1184	0.0869	0.0554

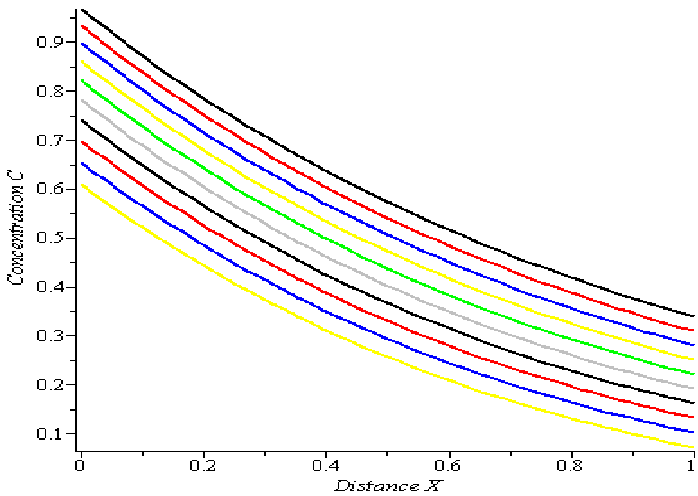

Fig 2: Represents concentration of contaminated or salt water $C(X,T)$ vs. distance X for auxiliary parameter $\hbar = 0.1$ and auxiliary function $H(Z,T) = 1$ [20] when T =0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 is fixed.

VI. CONCLUSION AND DISCUSSION

The equation (26) represents concentration of the contaminated or salt water for any distance X and time T >0. It is converges for embedding parameter $\varepsilon = 1$ and for auxiliary parameter $\hbar = 0.1$ which is expressed in term of negative exponential term of X and time $T > 0$. Concentration C will be one from guess value of the exact solution for $X = 0$, T= 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. Fig 2 represents the solution for concentration C vs. distance X shows that concentration of the contaminated or salt water is decreasing as distance X increasing for $T > 0$. From fig. 2 it can conclude that for $T = 0.1$ concentration of contaminated or salt water is decreasing as distance X increasing and when time is increasing and due to different deformation added to C, the concentration of contaminated or salt water is successively decreasing exponentially. Since the equation (7) is diffusion type Burger's equation for longitudinal dispersion phenomenon. Hence solution is graphically as well as physically consistent with phenomenon. This is physically fact with phenomenon of the longitudinal dispersion of contaminated or salt water in homogenous porous medium.

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