The Bloch Space of Analytic functions

S. Nagendra¹, Prof. E. Keshava Reddy²
¹Department of Mathematics, Government Degree College, Porumamilla
²Department of Mathematics, JNTUA

Abstract: We shall state and prove a characterization for the Bloch space and obtain analogous characterization for the little Bloch space of analytic functions on the unit disk in the complex plane. We shall also state and prove three containment results related to Bloch space and Little Bloch space.

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I. INTRODUCTION

We let \( D = \{ z \in \mathbb{C} / |z| < 1 \} \)

For \( w \in D \), the Möbius transformation \( \phi_w \) is defined by

\[
\phi_w(z) = \frac{w-z}{1-w\bar{z}} \quad \text{for} \quad z \in D
\]

Then

\[
1 - |\phi_w(z)|^2 = 1 - \phi_w(z) \overline{\phi_w(z)}
= 1 - \left( \frac{w-z}{1-w\bar{z}} \right) \left( \frac{\overline{w}-\overline{z}}{1-\overline{w}z} \right)
= 1 - \left( \frac{|w|^2}{|1-w\bar{z}|^2} \right) \left( 1 - |z|^2 \right) \quad - (1)
\]

So, the function \( \phi_w \) maps \( D \) on to itself and \( \partial D \) on to itself. It is easy to verify that \( \phi_w \) is its own inverse. Noting that \( \phi_w^i(z) = \frac{|w|^2 - 1}{(1-\overline{w}z)^2} \), the above identity states:

\[
\left( 1 - |z|^2 \right) |\phi_w^i(z)| = 1 - |\phi_w(z)|^2 \quad - (2)
\]

Bloch space \( B \) is the space of all analytic functions \( f \) on \( D \) for which

\[
\text{Sup}_{z \in D} \left( 1 - |z|^2 \right) |f^i(z)| < \infty
\]

and \( B \) becomes a Banach space with respect to the semi norm

\[
\|f\|_B = \text{Sup}_{z \in D} \left( 1 - |z|^2 \right) |f^i(z)|
\]
Using (2), we have

\[ \| f \phi_w \|_B = \sup_{z \in D} \left| (1-|z|^2) \left( f \phi_w \right) (z) \right| \]

\[ = \sup_{z \in D} \left| f^i \phi_w(z) \right| |\phi_w(z)| \]

\[ = \sup_{\phi_w(z) \neq D} \left| (1-|\phi_w(z)|^2) \left| f^i \phi_w(z) \right| \right| \]

\[ = \| f \|_B \]

\[ \Rightarrow \| f \phi_w \|_B = \| f \|_B \quad - (3) \]

Thus Bloch space is a Mobius invariant space.

In the next section, we shall state and prove a criterion for containment in the Bloch space and little Bloch space.

### II. Characterization For Bloch And Little Bloch Space

**A. Theorem 1**

For an analytic function \( f \) on \( D \)

\[ f \in B \iff \sup \left\{ \frac{(1-|z|^2)(1-|w|^2)}{1-wz} \left| f(z) - f(w) \right| : z, w \in D, z \neq w \right\} < \infty \]

**Proof:** Suppose for an analytic function \( f \) on \( D \)

\[ \sup \left\{ \frac{(1-|z|^2)(1-|w|^2)}{1-wz} \left| f(z) - f(w) \right| : z, w \in D, z \neq w \right\} < \infty \]

\[ \Rightarrow \frac{(1-|z|^2)(1-|w|^2)}{1-wz} \left| f(z) - f(w) \right| < \infty, \forall z, w \in D, z \neq w \]

Taking limit as \( w \to z \), we get

\[ \sup_{z \in D} \left( \frac{1-|z|^2}{1-|z|^2} \right) \left| f^i(z) \right| < \infty \]

\[ \Rightarrow \sup_{z \in D} \left| f^i(z) \right| < \infty \]

\[ \Rightarrow f \in B \]

For the next part, suppose \( f \in B \)

\[ \Rightarrow \sup_{z \in D} \left( 1-|z|^2 \right) \left| f^i(z) \right| < \infty \]

\[ \Rightarrow \left( 1-|z|^2 \right) \left| f^i(z) \right| < \| f \|_B, \forall z \in D \quad - (4) \]
Then for each \( u \in D \), we have

\[
(f(u) - f(0)) = \int_0^1 f'(tu) u dt
\]

\[
\Rightarrow |f(u) - f(0)| = \left| \int_0^1 f'(tu) u dt \right|
\]

\[
\leq \int_0^1 |f'(tu)| |u| dt
\]

\[
\leq \int_0^1 \|f\|_{L^1} |u| dt
\]

\[
\leq \int_0^1 \|f\|_{L^1} \left( 1 - t^2 |u|^2 \right) |u| dt
\]

\[
\leq \|f\|_{L^1} \frac{\log(1 - |u|)}{1 - |u|}
\]

Now for \( z, w \in D \) replace \( f \) in the above inequality by \( fo \phi_w \) and let \( u = \phi_w(z) \). Using \( \phi_w(\phi_w(z)) = z \) and identities (1) and (3) we have

\[
\|fo \phi_w\|_{L^1} \leq 2|u| |1 - |u||^2, \quad \forall \ u \in D
\]
We briefly discuss the little Bloch space $B_0$. The set of all analytic functions $f$ on $D$ for which

$$\lim_{|z| \to 1} (1-|z|^2)|f^1(z)| = 0$$

For an analytic function $f$ on $D$ and $0 < t < 1$ the dilate $f_t$ is the function defined by $f_t(z) = f(tz)$. It is known that for an analytic function $f$ on $D$:

$$f \in B_0 \iff \|f - f_t\|_B \to 0 \text{ as } t \to 1$$

Hence,

$$\lim_{|z| \to 1} \left| \frac{1-|z|^2}{1-|w|^2} \right|\left| \frac{f(z) - f(w)}{z - w} \right| \leq 2 \|f\|_B - (5)$$

In analogy to theorem (1), we have the following result.

**B. THEOREM 2**

For an analytic function $f$ on $D$:

$$\lim \sup_{|z| \to 1} \left\{ \frac{(1-|z|^2)(1-|w|^2)}{|1-wz|^2} \left| \frac{f(z) - f(w)}{z - w} \right| : z, w \in D, z \neq w \right\} = 0$$

Proof:

Taking limit as $w \to z$ in the condition of the statement, we get

$$\lim \sup_{|z| \to 1} \left[ \frac{(1-|z|^2)^2}{|1-z|^2} \left| f^1(z) \right| \right] = 0$$

Hence,

$$\lim \sup_{|z| \to 1} \left(1-|z|^2\right)|f^1(z)| = 0$$

$$\Rightarrow f \in B_0$$

Suppose $f \in B_0$, then $f \cdot f_t \in B$

Applying inequality (5) for $f_t \in B$, we have
Applying inequality (5) for \( f - f_t \in \mathbb{B} \), we have
\[
\frac{1-|f|^2}{1-|w|^2} \left| \frac{f(z) - f(w)}{z-w} \right| \leq 2\|f - f_t\|_B - (7)
\]
Inequality (6), (7) and triangle inequality imply that
\[
\frac{1-|f|^2}{1-|w|^2} \left| \frac{f(z) - f(w)}{z-w} \right| \leq \frac{1-|f|^2}{1-|w|^2} \left| \frac{(f-f_t)(z)-(f-f_t)(w)}{z-w} \right| + \frac{2|f|^2}{1-|w|^2} \left| \frac{f_t(z) - f_t(w)}{z-w} \right|
\]
\[
\leq \frac{1-|f|^2}{1-|w|^2} \left| \frac{(f-f_t)(z)-(f-f_t)(w)}{z-w} \right| + \frac{2|f|^2}{1-|w|^2} \left| \frac{f_t(z) - f_t(w)}{z-w} \right|
\]
Now first letting \( |z| \to 1 \) and then \( t \to 1 \), we get
\[
\lim_{|z| \to 1} \sup |f(z) - f(w)| : z, w \in D, z \neq w = 0
\]
In the next section, we shall prove three results related to containment of Bloch and Little Bloch space

### III. Containment Results of Bloch and Little Bloch Space

Let \( \phi \) be a bounded analytic function on \( D \), then there exists a constant
\( M > 0 \) such that \( \|\phi(z)\| \leq M, \forall z \in D \)
From Cauchy’s integral formula, we have
\[
\phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(w)dw}{(w-z)^2}
\]
where $C$ is any closed disc of radius $r$ in neighbourhood of 1 and containing $z$, then $|\phi(z)| \leq \frac{4M}{r}$ in the concentric disc of radius $\frac{r}{2}$. This implies that $\phi(z)$ is bounded in any neighbourhood of 1 contained in $D$ whenever so is $\phi(z)$.

A. THEOREM 3
If $f \in B$ then $f + k \in B$ where $k \in C$ is a constant
Proof: It is very easy to see that
$$f'(z) = (f + k)'(z)$$
Therefore $\sup_{z \in D} \left(1 - |z|^2\right)|f'(z)| = \sup_{z \in D} \left(1 - |z|^2\right)|(f + k)'(z)|$
Hence $f + k \in B$ whenever $f \in B$

B. THEOREM 4
If $f \in B_0$ is bounded and $\phi$ is any bounded and analytic function on $D$ then $\phi f \in B_0$.
Proof: From the definition of $B_0$, $\lim_{|z| \to 1^-} \left(1 - |z|^2\right)|f'(z)| = 0$
$$\phi f(z) = \phi(z)f'(z) + f(z)\phi(z)$$
Note that
$$\Rightarrow \left|\phi f(z)\right| \leq \left|\phi(z)\right|\left|f'(z)\right| + \left|f(z)\right|\left|\phi(z)\right|$$
$$\Rightarrow \left(1 - |z|^2\right)\left|\phi f(z)\right| \leq \left(1 - |z|^2\right)\left|\phi(z)\right|\left|f'(z)\right| + \left(1 - |z|^2\right)\left|f(z)\right|\left|\phi(z)\right|$$
Taking limit as $|z| \to 1^-$, the first term on RHS tends to 0 because of the hypothesis and $\phi$ is bounded and the second term since $f'$ and $\phi'$ bounded in the neighbourhood of 1 as $\phi$ is bounded on $D$ tends to 0.
Hence $\lim_{|z| \to 1^-} \left(1 - |z|^2\right)\left|\phi f(z)\right| = 0$
Therefore $\phi f \in B_0$.

C. THEOREM 5
If $f, g$ are bounded functions of $B_0$, then $fg \in B_0$.
Proof: From the definition of $B_0$,
$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right)\left|f'(z)\right| = 0$$
$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right)\left|g'(z)\right| = 0$$
Note that
$$0 < \left(1 - |z|^2\right)\left|fg'(z)\right| \leq \left(1 - |z|^2\right)\left|f'(z)\right|\left|g(z)\right| + \left(1 - |z|^2\right)\left|f(z)\right|\left|g'(z)\right|$$
Taking limit as $|z| \to 1^-$, we get
$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right)\left|fg'(z)\right| = 0$$
\therefore $fg \in B_0$. 


IV. CONCLUSION

I invite interested readers to pursue geometric interpretation of characterization theorems that we proved in this paper and also similar containment results related to the Bloch space.

REFERENCES