On $\pi$Gr-Separation Axioms

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Abstract: In the present paper, we introduce and study the concept of $\pi gr$-$T_i$ space (for $i=0,1,2$) and obtain the characterization of $\pi gr$-regular space, $\pi gr$-normal space by using the notion of $\pi gr$-open sets. Further, some of their properties and results are discussed.

Key Words: $\pi gr$-$T_i$-space, $\pi gr$-$T_i$-space, $\pi gr$-$T_2$-space, $\pi gr$-normal, $\pi gr$-regular.

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I. Introduction


The purpose of this paper is to introduce and study $\pi gr$-separation axioms in topological spaces. Further we introduced the concepts of $\pi gr$-regular space, $\pi gr$-Normal Space and study their behaviour.

II. Preliminaries

Throughout this paper $(X,\tau)$, $(Y,\sigma)$ (or simply $X$, $Y$) always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

For a subset A of a topological space X, the closure and interior of A with respect to $\tau$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$ respectively.

Definition 2.1
A subset $A$ of $X$ is said to be regular open [12] if $A=\text{int}(\text{cl}(A))$ and its complement is regular closed.

The finite union of regular open set is $\pi$-open set[6,14] and its complement is $\pi$-closed set. The union of all regular open sets contained in $A$ is called $\text{rint}(A)$[regular interior of $A$] and the intersection of regular closed sets containing $A$ is called $\text{rcl}(A)$[regular closure of $A$]

Definition 2.2
A subset $A$ of $X$ is called $\pi gr$-closed[7] if $\text{rcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open,. The complement of $\pi gr$-closed set is $\pi gr$-open set. The family of all $\pi gr$-closed subsets of $X$ is denoted by $\pi \text{GRC}(X)$ and $\pi gr$-open subsets of $X$ is denoted by $\pi \text{GRO}(X)$

Definition 2.3
The intersection of all $\pi gr$-closed containing a set $A$ is called $\pi gr$-closure of $A$ and is denoted by $\pi gr$,Cl$(A)$. The union of $\pi gr$-open sets contained in $A$ is called $\pi gr$-interior of $A$ and is denoted by $\pi gr$,int$(A)$.

Definition 2.4
A function $f:(X,\tau)\rightarrow (Y,\sigma)$ is called

1. Continuous [9] if $f^{-1}(V)$ is closed in $X$ for every closed set $V$ in $Y$.
2. Regular continuous ([r-continuous]) [3] if $f^{-1}(V)$ is regular-closed in $X$ for every closed set $V$ in $Y$.
3. An R-map[6] if $f^{-1}(V)$ is regular closed in $X$ for every regular closed set $V$ of $Y$.
4. $\pi gr$-continuous[7] if $f^{-1}(V)$ is $\pi gr$-closed in $X$ for every closed set $V$ in $Y$.
5. $\pi gr$-irresolute[7] if $f^{-1}(V)$ is $\pi gr$-closed in $X$ for every $\pi gr$-closed set $V$ in $Y$. 
Definition: 2.5
A space X is called a πgr-T_{1/2} space [7] if every πgr-closed set is regular closed.

Definition: 2.6
A map f: X→Y is called
1. Closed [9] if f(U) is Y for every closed set U of X.
2. R-closed (i.e., regular closed) [12] if f(U) is regular closed in Y for every closed set U of X.
3. rc-preserving [6] if f(U) is regular closed in Y for every regular closed set U of X.

Definition: 2.7
A map f: X→Y is called
1. πgr-open map if f(V) is πgr-open in Y for every open set V in X.
2. strongly πgr-open map (M-πgr-open) if f(V) is πgr-open in Y for every πgr-open set V in X.
3. Quasi πgr-open if f(V) is open in Y for every πgr-open set V in X.
4. Almost πgr-open map if f(V) is πgr-open in Y for every regular open set V in X.

Definition: 2.8
A space X is said to be R-regular [10] if for each closed set F and each point x∈F, there exists disjoint regular open sets U and V such that x∈U and F⊂V.

Definition: 2.9
A space X is said to be R-Normal [11,13] (Mildly Normal) if for every pair of disjoint regular closed sets E and F of X, there exists disjoint open sets U and V such that E⊂U and F⊂V.

III. πGr Separation Axioms

In this section, we introduce and study πgr-separation axioms and obtain some of its properties.

Definition: 3.1
A space X is said to be πgr-T_0-space if for each pair of distinct points x and y of X, there exists a πgr-open set containing one point but not the other.

Theorem: 3.2
A space X is πgr-T_0-space if and only if πgr-closures of distinct points are distinct.
Proof: Let x and y be distinct points of X. Since X is a πgr-T_0-space, there exists a πgr-open set G such that x∈G and y∉G. Consequently, X−G is a πgr-closed set containing y but not x. But πgr-cl(y) is the intersection of all πgr-closed sets containing y. Hence y∈πgr-cl(y), but x∉πgr-cl(y) as x∉X−G. Therefore, πgr-cl(x)≠πgr-cl(y).
Conversely, let πgr-cl(x)=πgr-cl(y) for x≠y.
Then there exists at least one point z∈X such that z∉πgr-cl(y).
We have to prove x∉πgr-cl(y), because if x∈πgr-cl(y), then {x}⊂πgr-cl(y)
⇒ πgr-cl(x)=πgr-cl(y). So, z∉πgr-cl(y), which is a contradiction. Hence x∉πgr-cl(y) ⇒ x∈X−πgr-cl(y),
which is a πgr-open set containing x but not y. Hence X is a πgr-T_0-space.

Theorem: 3.3
If f:X→Y is a bijection, strongly-πgr-open and X is a πgr-T_0-space, then Y is also πgr-T_0-space.
Proof: Let y_1 and y_2 be two distinct points of Y. Since f is bijective, there exists points x_1 and x_2 of X such that f(x_1) = y_1 and f(x_2) = y_2. Since X is a πgr-T_0-space, there exists a πgr-open set G such that x_1∈G and x_2∉G. Therefore, y_1 = f(x_1) ∈ f(G), y_2 = f(x_2)∉ f(G). Since f is strongly πgr-open function, f(G) is πgr-open in Y. Thus, there exists a πgr-open set f(G) in Y such that y_1 ∈ f(G) and y_2 ∉ πgr-T_0-space.

Definition: 3.4
A space X is said to be πgr-T_1-space if for any pair of distinct points x and y, there exists πgr-open sets G and H such that x∈G, y∉G and x∉H, y∈H.

Theorem: 3.5
A space X is πgr-T_1-space iff singletons are πgr-closed sets.
Proof: Let X be a πgr-T_1-space and x∈X. Let y∈X−{x}. Then for x≠y, there exists πgr-open set U_y such that y∈U_y and x∉U_y.
Conversely, y∈U_y ⇒ x∉U_y.
That is X−{x} = $\bigcup_{y \in X−\{x\}}$, which is πgr-open set.
Hence {x} is πgr-closed set.
Conversely, suppose \([x] \in \pi\text{-gr-closed set for every } x \in X\). Let \(x, y \in X\) with \(x \neq y\). Now, \(x \neq y \Rightarrow y \in X \setminus \{x\}\). Hence \(X \setminus \{x\}\) is \(\pi\text{-gr-open set containing } y\) but not \(x\). Similarly, \(X \setminus \{y\}\) is \(\pi\text{-gr-open set containing } x\) but not \(y\). Therefore, \(X\) is a \(\pi\text{-gr-T}_1\) space.

**Theorem 3.6**

If \(f : X \rightarrow Y\) is strongly \(\pi\text{-gr-open bijective map and } X\) is \(\pi\text{-gr-T}_1\) space, then \(Y\) is \(\pi\text{-gr-T}_1\) space.

**Proof:** Let \(f : X \rightarrow Y\) be bijective and strongly-\(\pi\text{-gr-open function. Let } X\) be a \(\pi\text{-gr-T}_1\) space and \(y_1, y_2\) be any two distinct points of \(Y\).

Since \(f\) is bijective, there exists distinct points \(x_1, x_2\) of \(X\) such that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\). Now, \(X\) being a \(\pi\text{-gr-T}_1\) space, there exists \(\pi\text{-gr-open sets } G\) and \(H\) such that \(x_1 \in G, x_2 \notin G\) and \(x_1 \notin H, x_2 \in H\). Since \(y_1 = f(x_1) \in f(G)\) but \(y_2 = f(x_2) \notin f(G)\) and \(y_2 = f(x_2) \notin f(H)\), and \(y_1 \notin f(H)\).

Now, \(f\) being strongly-\(\pi\text{-gr-open}, f(G)\) and \(f(H)\) are \(\pi\text{-gr-open subsets of } Y\) such that \(y_1 \notin f(G)\) but \(y_2 \notin f(G)\) and \(y_2 \notin f(H)\). Hence \(Y\) is \(\pi\text{-gr-T}_1\)-space.

**Theorem 3.7**

If \(f : X \rightarrow Y\) is \(\pi\text{-gr-continuous injection and } Y\) is \(T_1\), then \(X\) is \(\pi\text{-gr-T}_1\) space.

**Proof:** Let \(f : X \rightarrow Y\) be \(\pi\text{-gr-continuous injection and } Y\) be \(T_1\). For any two distinct point \(x_1, x_2\) of \(X\), there exists distinct points \(y_1, y_2\) of \(Y\) such that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\).

Since \(Y\) is \(T_1\)-space, there exists open sets \(U\) and \(V\) in \(Y\) such that \(y_1 \in U\) and \(y_2 \notin U\) and \(y_1 \notin V\) and \(y_2 \in V\).

i.e. \(\begin{aligned} x_1 & \in f^{-1}(U), x_1 \notin f^{-1}(V) \quad \text{and} \quad x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U) \end{aligned}\)

Since \(f\) is \(\pi\text{-gr-continuous}, f^{-1}(U), f^{-1}(V)\) are \(\pi\text{-gr-open sets in } X\).

Thus for two distinct points \(x_1, x_2\) of \(X\), there exists \(\pi\text{-gr-open sets } f^{-1}(U)\) and \(f^{-1}(V)\) such that \(x_1 \in f^{-1}(U), x_2 \notin f^{-1}(V)\) and \(x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)\).

Therefore, \(X\) is \(\pi\text{-gr-T}_1\) space.

**Theorem 3.8**

If \(f : X \rightarrow Y\) be \(\pi\text{-gr-irresolute function, and } Y\) is \(\pi\text{-gr-T}_1\) space, then \(X\) is \(\pi\text{-gr-T}_1\) space.

**Proof:** Let \(x_1, x_2\) be distinct points in \(X\). Since \(f\) in injective, there exists distinct points \(y_1, y_2\) of \(Y\) such that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\).

Since \(Y\) is \(\pi\text{-gr-T}_1\) -space, there exists \(\pi\text{-gr-open sets } U\) and \(V\) in \(Y\) such that \(y_1 \in U\) and \(y_2 \notin U\) and \(y_1 \notin V\) and \(y_2 \in V\).

i.e. \(\begin{aligned} x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V) \quad \text{and} \quad x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U) \end{aligned}\)

Since \(f\) is \(\pi\text{-gr-irresolute}, f^{-1}(U), f^{-1}(V)\) are \(\pi\text{-gr-open sets in } X\).

Thus for two distinct points \(x_1, x_2\) of \(X\), there exists \(\pi\text{-gr-open sets } f^{-1}(U)\) and \(f^{-1}(V)\) such that \(x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)\) and \(x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)\).

Hence \(X\) is \(\pi\text{-gr-T}_1\) space.

**Definition 3.9**

A space \(X\) is said to be \(\pi\text{-gr-T}_2\)-space, if for any pair of distinct points \(x, y\), there exists disjoint \(\pi\text{-gr-open sets } G\) and \(H\) such that \(x \in G\) and \(y \in H\).

**Theorem 3.10**

If \(f : X \rightarrow Y\) be \(\pi\text{-gr-continuous function, and } Y\) is \(T_2\)-space, then \(X\) is \(\pi\text{-gr-T}_2\)-space.

**Proof:** Let \(f : X \rightarrow Y\) be \(\pi\text{-gr-continuous function, and } Y\) be \(T_2\). For any two distinct points \(x_1, x_2\) of \(X\), there exists distinct points \(y_1, y_2\) of \(Y\) such that \(y_1 = f(x_1), y_2 = f(x_2)\). Since \(Y\) is \(T_2\)-space, there exists disjoint open sets \(U\) and \(V\) in \(Y\) such that \(y_1 \in U\) and \(y_2 \in V\).

i.e. \(\begin{aligned} x_1 \in f^{-1}(U), x_1 \in f^{-1}(V) \quad \text{and} \quad x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U) \end{aligned}\)

Since \(f\) is \(\pi\text{-gr-continuous}, f^{-1}(U)\) and \(f^{-1}(V)\) are \(\pi\text{-gr-open sets in } X\).

Further \(f\) is injective, \(f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi\).

Thus, for two disjoint points \(x_1, x_2\) of \(X\), there exists disjoint \(\pi\text{-gr-open sets } f^{-1}(U)\) and \(f^{-1}(V)\) such that \(x_1 \in f^{-1}(U)\) and \(x_2 \in f^{-1}(V)\). Hence \(X\) is \(\pi\text{-gr-T}_2\)-space.

**Theorem 3.11**

If \(f : X \rightarrow Y\) be \(\pi\text{-gr-irresolute injective function and } Y\) is \(\pi\text{-gr-T}_2\)-space, then \(X\) is \(\pi\text{-gr-T}_2\)-space.

**Proof:** Let \(x_1, x_2\) be any two distinct points in \(X\). Since \(f\) in injective, there exists distinct points \(y_1, y_2\) of \(Y\) such that \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\).

Since \(Y\) is \(\pi\text{-gr-T}_2\), there exist disjoint \(\pi\text{-gr-open sets } U\) and \(V\) in \(Y\) such that \(y_1 \in U\) and \(y_2 \in V\).

i.e. \(\begin{aligned} x_1 \in f^{-1}(U), x_1 \in f^{-1}(V) \quad \text{and} \quad x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U) \end{aligned}\)

Since \(f\) is \(\pi\text{-gr-irresolute injective, } f^{-1}(U)\) and \(f^{-1}(V)\) are disjoint \(\pi\text{-gr-open sets in } X\).

Thus, for two distinct points \(x_1, x_2\) of \(X\), there exists disjoint \(\pi\text{-gr-open sets } f^{-1}(U)\) and \(f^{-1}(V)\) such that \(x_1 \in f^{-1}(U)\) and \(x_2 \in f^{-1}(V)\).
Hence X is πgr-T_{2}\text{-space}.

**Theorem: 3.12**

In any topological space, the following are equivalent.

1. X is πgr-T_{2}\text{-space}.
2. For each x ≠ y, there exists a πgr-open set U such that x ∈ U & y ∉ πgr-cl(U)
3. For each x ∈ X, \{x\} = \cap \{πgr -cl(U): U is a πgr - open set in Z is x ∈ U\}.

**Proof:** (1) ⇒ (2): Assume (1) holds.
Let x ∈ X and x ≠ y, then there exists disjoint πgr-open sets U and V such that x ∈ U and y ∈ V. Clearly, X–V is πgr-closed set. Since U ∩ V = ∅, U ⊂ X–V.

Therefore, πgr-cl(U) ⊂ πgr-cl(X–V)

Y ∈ X–V ⇒ y ∈ πgr-cl(X–V) and hence y ∉ πgr-cl(U), by the above argument.

(2) ⇒ (3): For each x ≠ y; there exists a πgr-open set U such that x ∈ U and y ∉ πgr-cl(U)

So, y ∉ \{πgr -cl(U): U is a πgr - open set in X and x ∈ U\} = \{x\}.

(3) ⇒ (1): Let x, y ∈ X and x ≠ y.

By hypothesis, there exists a πgr-open set U such that x ∈ U and y ∉ πgr-cl(U).

⇒ There exists a πgr-closed set V set y ∈ V. Therefore, y ∈ X–V and X–V is a πgr -open set.

Thus, there exists two disjoint πgr-open sets U and X–V such that x ∈ U and y ∈ X–V.

Therefore, X is πgr-T_{2}\text{-space}.

### IV. πGr- Regular Space

**Definition: 4.1**

A space X is said to be πgr-regular if for each closed set F and each point x ∉ F, there exists disjoint πgr-open sets U and V such that x ∈ U and F ⊂ V.

**Theorem: 4.2**

Every πgr-regular T_{0} - space is πgr-T_{2}.

**Proof:** Let x, y ∈ X such that x ≠ y.

Let X be a T_{0}-space and V be an open set which contains x but not y.

Then X–V is a closed set containing y but not x. Now, by πgr-regularity of X, there exists disjoint πgr-open sets U and W such that x ∈ U and X–V ⊂ W.

Since y ∈ X–V, y ∈ W.

Thus, for x, y ∈ X with x ≠ y there exists disjoint open sets U and W such that x ∈ U and y ∈ W.

Hence X is πgr-T_{2}\text{-space}.

**Theorem: 4.3**

If f: X → Y is continuous bijective, πgr- open function and X is a regular space, then Y is πgr-regular.

**Proof:** Let Y be a closed set in Y and y ∉ F. Take y = f(x) for some x ∈ X.

Since f is continuous, f^{−1}(F) is closed set in X such that x ∉ f^{−1}(F). (since f(x) ∉ F)

Now, X is regular, there exists disjoint open sets U and V such that x ∈ U and f^{−1}(F) ⊂ V.

i.e. y = f(x) ∈ f(U) and F ⊂ f(V).

Since f is πgr-open function, f(U) and f(V) are πgr-open sets in Y.

Since f is bijective, f(U) ∩ f(V) = f(U ∩ V) = f(∅) = ∅.

⇒ Y is πgr-regular.

**Theorem: 4.4**

If f: X → Y is regular continuous bijective, almost πgr-open function and X is R-regular space, then Y is πgr-regular.

**Proof:**

Let Y be a closed set in Y and y ∉ F.

Take x = f(y) for some x ∈ X.

Since f is regular continuous function, f^{−1}(F) is regular closed in X and hence closed in X.

⇒ x = f^{−1}(y) ∈ f^{−1}(F).

Now, X is R-regular, there exists disjoint regular open sets U and V such that x ∈ U and f^{−1}(F) ⊂ V.

i.e. y = f(x) ∈ f(U) and F ⊂ f(V).

Since f is almost πgr-open function f(U) and f(V) are πgr-open sets in Y and also f is bijective, f(U) ∩ f(V) = f(U ∩ V) = f(∅) = ∅.

⇒ Y is πgr-regular.
Theorem: 4.5
If f: X→Y is continuous, bijective, strongly πgr-open function (quasi πgr-open) and X is πgr-regular space, then Y is πgr-regular (regular).

Proof: Let F be a closed set in Y and y ∈ F.
Take y=f(x) for some x ∈ X.
Since f is continuous bijective, f^(-1)(F) is closed in X and x ∉ f^(-1)(F).
Now, since X is πgr-regular, there exists disjoint πgr-open sets U and V such that x ∈ U and f^(-1)(F) ⊆ V.

Theorem: 4.6
If f:X→Y is πgr-continuous, closed, injection and Y is regular, then X is πgr-regular.

Proof: Let F be a closed in X and x ∈ F.
Since f is closed injection, f(F) is closed set in Y such that f(x) ∉ f(F).
Now, Y is regular, there exists disjoint open sets G and H such that f(x) ∈ G and f(F) ⊆ H.
This implies x ∈ f^(-1)(G) and F ⊆ f^(-1)(H).
Since f is πgr-continuous, f^(-1)(G) and f^(-1)(H) are πgr-open sets in X.
Further, f^(-1)(G) ∩ f^(-1)(H) = ∅.
Hence X is πgr-regular.

Theorem: 4.7
If f:X→Y is almost πgr- continuous, closed injection and Y is R-regular, then X is πgr-regular.

Proof: Let F be a closed set in X and x ∉ F. Since f is closed injection, f(F) is closed set in Y such that f(x) ∉ f(F).
Now, Y is R-regular, there exists disjoint regular open sets G and H such that f(x) ∈ G and f(F) ⊆ H.
⇒ x ∈ f^(-1)(G) & F ⊆ f^(-1)(H).
Since f is almost πgr-continuous, f^(-1)(G) & f^(-1)(H) are πgr-open sets in X.
Further, f^(-1)(G) ∩ f^(-1)(H) = ∅.
Hence X is πgr-regular.

Theorem: 4.8
If f: X→Y is πgr-irresolute, closed, injection and Y is πgr-regular, then X is πgr-regular.

Proof: Let F be a closed set in X and x ∉ F. Since f is closed injection, f(F) is closed set in Y such that f(x) ∉ f(F).
Now, Y is πgr-regular, there exists disjoint πgr-open sets G and H such that f(x) ∈ G and f(F) ⊆ H.
⇒ x ∈ f^(-1)(G) & F ⊆ f^(-1)(H).
Since X is πgr-irresolute, f^(-1)(G) and f^(-1)(H) are πgr-open sets in X.
Further, f^(-1)(G) ∩ f^(-1)(H) = ∅ and hence X is πgr-regular.

Definition: 5.1
A space X is said to be πgr-Normal if for every pair of disjoint closed sets E & F of X, there exists disjoint πgr-open sets U & V such that E ⊆ U and F ⊆ V.

Theorem: 5.2
The following statements are equivalent for a Topological space X:
1. X is πgr- normal.
2. For each closed set A and for each open set U containing A, there exists a πgr-open set V containing A such that πgr-cl(V) ⊆ U.
3. For each pair of disjoint closed sets A and B, there exists πgr-open set U containing A such that πgr-cl(U)∩B = ∅.

Proof: (1)⇒(2): Let A be closed set and U be an open set containing A.
Then A ∩ (X−U) = ∅ and therefore they are disjoint closed sets in X.
Since X is πgr-normal, there exists disjoint πgr-open sets V and W such that A ⊆ V, X−U ⊆ W. i.e. X−W ⊆ U.
Now, ∀W ⊆ X−W implies V ⊆ X−W. Therefore, πgr-cl(V) ⊆ πgr-cl(X−W) = X−W, Because X−W is πgr-closed set.
Thus, A ⊆ V ∩ πgr-cl(V) ⊆ X−W ⊆ U.
Hence A ⊆ V ⊆ πgr-cl(V) ⊆ U.
(2)⇒(3): Let A and B be disjoint closed sets in X, then A ⊆ X−B and X−B is an open set containing A. By hypothesis, there exists a πgr-open set U such that A ⊆ U and πgr-cl(U) ⊆ X−B, which implies πgr-cl(U)∩B = ∅.
(3)⇒(1): Let A and B be disjoint closed sets in X. By hypothesis (3), there exists a πgr-open set U such that
A⊂U and πgr-cl(U)∩B = ø (or B⊂X − πgr-cl(B)).
Now, U and X− πgr-cl(U) are disjoint πgr-open sets such that A⊂U and B⊂X − πgr-cl(U).
Hence X is πgr-normal.

Definition 5.3
A space X is said to be mildly πgr-Normal if for every pair of disjoint regular closed sets E & F of X, there
exists disjoint πgr-open sets U & V such that E⊂U and F⊂V.

Theorem 5.4
If f:X→Y is continuous bijective, πgr-open function from a normal spaces X onto a space Y, then Y is πgr-
normal.

Proof: Let E and F be disjoint closed sets in Y,
Since f is continuous bijective f⁻¹(E) and f⁻¹(F) are disjoint closed sets in X.
Now, X is normal, there exists disjoint open sets U and V such that f⁻¹(E)⊂U, f⁻¹(F)⊂V.
i.e. E⊂f(U), f(U)⊂f(V).
Since f is πgr-open function, f(U) and f(V) are πgr-open sets in Y and f is injective, f(U)∩f(V) = f(U∩V) = f(ϕ)
⇒ϕ.Hence Y is πgr-Normal.

Theorem 5.5
If f:X→Y is regular continuous bijective, almost πgr-open function from a mildly normal space X onto a
space Y, then Y is πgr-normal.

Proof: Let E and F be disjoint closed sets in Y, Since f is regular continuous bijective f⁻¹(E) and
f⁻¹(F) are disjoint regular closed sets in X.
Now, X is mildly normal, there exists disjoint regular open sets U and V, such that f⁻¹(E)⊂U,
f⁻¹(F)⊂V.
i.e. E⊂f(U), F⊂f(V).Since f is almost πgr-open function, f(U) & f(V) are πgr-open sets in Y and f is injective,
f(U)∩f(V) = f(U∩V)
⇒ϕ.Hence Y is πgr-Normal.

Thus, Y is πgr-Normal.

Theorem 5.6
If f:X→Y is πgr-continuous, closed, bijective, and Y is normal, then X is πgr-normal.

Proof: Let E and F be disjoint closed sets in X, since f is closed injection, f(E) and f(F) are disjoint closed sets in
Y.
Now Y is normal, there exists disjoint open sets G and H such that f(E)⊂G, f(F)⊂H.
⇒ E⊂f⁻¹(G) & F⊂f⁻¹(H).
Since f is πgr-continuous, f⁻¹(G) and f⁻¹(H) are πgr-open sets in X.
Further, f⁻¹(G)∩f⁻¹(H) = ϕ.Hence X is πgr-Normal.

Theorem 5.7
If f:X→Y is almost πgr-continuous, R-closed injective, and Y is R-normal, then X is πgr-normal.

Proof: Let E and F be disjoint closed sets in Y. Since f is R-closed injection, f(E) and f(F) are disjoint regular
closed sets in Y.
Now Y is Mildly Normal,(i.e, R-normal), there exists disjoint regular open sets G and H such that f(E)⊂G, f(F)⊂H.
⇒ E⊂f⁻¹(G) & F⊂f⁻¹(H).
Since f is almost πgr-continuous, f⁻¹(G) and f⁻¹(H) are πgr-open sets in X.
Further, f⁻¹(G)∩f⁻¹(H) = ϕ.
Hence X is πgr-Normal.

Theorem 5.8
If f:X→Y is almost πgr-irresolute, R-closed injection, and Y is πgr-normal, then X is πgr-normal.

Proof: Let E and F be disjoint closed sets in Y. Since f is R-closed injection, f(E) and f(F) are disjoint regular
closed sets in Y.
Now Y is πgr-Normal, there exists disjoint πgr-open sets G and H such that f(E)⊂G, f(F)⊂H.
This implies E⊂f⁻¹(G) and F⊂f⁻¹(H).
Since f is πgr-irresolute, f⁻¹(G) and f⁻¹(H) are πgr-open sets in X.
Further, f⁻¹(G)∩f⁻¹(H) = ϕ.
⇒ X is πgr-Normal.
Theorem: 5.9
If \( f: X \to Y \) is continuous, bijective, \( M-\pi gr\)-open (quasi \( \pi gr\)-open) function from a \( \pi gr\)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( \pi gr\)-normal (normal).

**Proof:** Let \( E \) and \( H \) be disjoint closed sets in \( Y \). Since \( f \) is continuous bijective, \( f^{-1}(E) \) and \( f^{-1}(F) \) are disjoint closed sets in \( X \). Now, \( X \) is \( \pi gr\)-normal, there exists \( \pi gr\)-open sets \( U \) and \( V \) such that \( f^{-1}(E) \subseteq U \) and \( f^{-1}(F) \subseteq V \). That is \( E \subseteq f(U) \) and \( F \subseteq f(V) \). Since \( f \) is \( M-\pi gr\)-open (quasi \( \pi gr\)-open) function, \( f(U) \) and \( f(V) \) are \( \pi gr\)-open sets(open sets) in \( Y \) and \( f \) is bijective, \( f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset \).
Hence \( Y \) is \( \pi gr\)-normal (normal).

**BIBLIOGRAPHY**