On Some Integrals of Products of $H$-Functions

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Abstract: The object of the present paper is to evaluate an integral involving products of three $H$-function of different arguments which not only provides us the Laplace transform, Hankel transform ([8],p.3), Meijer’s Bessel transform ([8],p.121) and various other integral transforms of the product of two $H$-functions but also generalizes the result given earlier by many writers notably by Bailey ([3], p.38), Meijer ([8],p.422) and Slater ([20], p.54(3.7.2).

I. Introduction

The $H$-function occurring in the paper will be defined and represented by Inayat-Hussain [12] as follows:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N} \left[ z \mid (a_j, \alpha_j; A_j, a_j; b_j, \beta_j)_{M,j=1}^P \right] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \phi(\xi)z^\xi d\xi$$  \hspace{1cm} (1.1)

where

$$\phi(\xi) = \prod_{j=1}^{M} (\Gamma(b_j - \beta_j \xi))^{\frac{1}{\beta_j}} \prod_{j=1}^{N} \left\{ (1 - a_j + \alpha_j \xi)^{\frac{1}{a_j}} \right\}$$  \hspace{1cm} (1.2)

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j (j = 1, ..., P)$ and $b_j (j = 1, ..., Q)$ are complex parameters, $\alpha_j \geq 0 (j = 1, ..., P)$, $\beta_j \geq 0 (j = 1, ..., Q)$ (not all zero simultaneously) and exponents $A_j (j = 1, ..., N)$ and $B_j (j = N + 1, ..., Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $H$-function given by equation (1.1) have been given by (Buschman and Srivastava[6]).

$$\Omega = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} |A_j| - \sum_{j=M+1}^{P} |\beta_j B_j| - \sum_{j=N+1}^{Q} |\alpha_j| > 0$$  \hspace{1cm} (1.3)

and $|\arg(z)| < \frac{\pi}{2} \Omega$  \hspace{1cm} (1.4)

The behavior of the $H$-function for small values of $|z|$ follows easily from a result recently given by (Rathie [15],p.306,eq.(6.9)).

We have

$$H_{P,Q}^{M,N}[z] = 0 \left| (z) \right|, \gamma = \min_{1 \leq j \leq N} \left[ \text{Re} \left( \frac{b_j}{\beta_j} \right) \right], |z| \to 0$$  \hspace{1cm} (1.5)

If we take $A_j = 1 (j = 1, 2, ..., N), B_j = 1 (j = M + 1, ..., Q)$ in (1.1), the function $H_{P,Q}^{M,N}[.]$ reduces to the Fox’s $H$-function [9].

The following series representation for the $H$-function will be required in the sequel (see Rathie, [15]pp.305-306,eq.(6.8)):

$$H_{P,Q}^{M,N}[z \mid (a_j, \alpha_j; A_j, a_j; b_j, \beta_j)_{M,j=1}^P \mid (b_j, \beta_j)_{N,j=1}^Q] =$$
\[
\sum_{h=1}^{M} \sum_{i=0}^{N} \Gamma(b_j - \beta_j \xi_{h,r}) \prod_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \xi_{h,r})\}^{A_j} (-1)^r z^{S_h,r} \\
\prod_{j=M+1}^{p} \{\Gamma(1 - b_j + \beta_j \xi_{h,r})\}^{b_j} \prod_{j=N+1}^{p} \Gamma(a_j - \alpha_j \xi_{h,r}) r! \beta_h
\]

Where
\[
\xi_{h,r} = \frac{(b_h + r)}{\beta_h}.
\]

II. The \( \overline{H} \)-Function Of Two Variables

The \( \overline{H} \)-function of two variables will be defined and represented in the following manner:

\[
\overline{H}[x, y] = \overline{H}\left[\begin{array}{c} x \\ y \end{array}\right] = \overline{H}_{a,b}^{c,d} \left[ x \left(\begin{array}{c} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_m \end{array}\right) \right]
\]

\[
= -\frac{1}{4\pi^2} \int_{L_1}^{L_2} \int \phi_1(\xi, \eta) \phi_2(\xi) \phi_3(\eta) x^y d\xi d\eta
\]

Where
\[
\phi_1(\xi, \eta) = \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi + A_j \eta)
\]

\[
\phi_2(\xi) = \prod_{j=1}^{n} \Gamma(1 - c_j + \gamma_j \xi) \prod_{j=m+1}^{q_2} \Gamma(d_j - \delta_j \xi)
\]

\[
\phi_3(\eta) = \prod_{j=1}^{n} \Gamma(1 - e_j + E_j \eta) \prod_{j=m+1}^{q_1} \Gamma(f_j - F_j \eta)
\]

Where \( x \) and \( y \) are not equal to zero (real or complex), and an empty product is interpreted as unity \( p_1, q_1, n_1, m_1 \) are non-negative integers such that \( 0 \leq n_1 \leq p_1, o \leq m_1 \leq q_1 \) \( i = 1, 2, 3; j = 2, 3 \). All the \( a_j, b_j, c_j, d_j, e_j, f_j \) are complex parameters. \( \gamma_j \geq 0(j = 1, 2, \ldots, q_1); \delta_j \geq 0(j = 1, 2, \ldots, q_2) \) (not all zero simultaneously), similarly \( E_j \geq 0(j = 1, 2, \ldots, q_1); F_j \geq 0(j = 1, 2, \ldots, q_2) \) (not all zero simultaneously). The exponents \( K_j(j = 1, 2, \ldots, n_1), L_j(j = m_1 + 1, \ldots, q_2), R_j(j = 1, 2, \ldots, q_1), S_j(j = m_1 + 1, \ldots, q_2) \) can take on non-negative values.

The contour \( L_1 \) is in \( \xi \)-plane and runs from \( -i\infty \) to \( +i\infty \). The poles of \( \Gamma(1 - \delta_j \xi) \) \((j = 1, 2, \ldots, m_1) \) lie to the right and the poles of \( \Gamma(1 - \alpha_j \xi + A_j \eta) \) \((j = 1, 2, \ldots, n_1) \) to the left of the contour. For \( K_j(j = 1, 2, \ldots, n_1) \) not an integer, the poles of gamma functions of the numerator in (1.9) are converted to the branch points.
The contour $L_2$ is in $\eta$-plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma(f_j - F_j \eta)(j = 1, 2, \ldots, m_j)$ lie to the right and the poles of $\Gamma\left((1-e^{-j} + E_j \eta)\right)^{R_j}(j = 1, 2, \ldots, n_j)$, $\Gamma(1-a_j + \alpha_j \xi + A_j \eta)(j = 1, 2, \ldots, n_j)$ to the left of the contour. For $R_j(j = 1, 2, \ldots, n_j)$ not an integer, the poles of gamma functions of the numerator in (1.10) are converted to the branch points.

The functions defined in (1.7) is an analytic function of $x$ and $y$, if

$$U = \sum_{j=1}^{m_1} \alpha_j + \sum_{j=1}^{m_1} \beta_j - \sum_{j=1}^{m_1} \delta_j < 0$$

$$V = \sum_{j=1}^{n_1} A_j + \sum_{j=1}^{n_1} E_j - \sum_{j=1}^{n_1} B_j - \sum_{j=1}^{n_1} F_j < 0$$

The integral in (1.7) converges under the following set of conditions:

$$\Omega = \sum_{j=1}^{m_1} \alpha_j - \sum_{j=m_1+1}^{m_2} \alpha_j + \sum_{j=1}^{m_1} \delta_j - \sum_{j=m_1+1}^{m_2} \delta_j L_j + \sum_{j=1}^{m_2} \gamma_j K_j - \sum_{j=m_1+1}^{m_2} \gamma_j - \sum_{j=1}^{m_2} \beta_j > 0$$

$$\Lambda = \sum_{j=1}^{n_1} A_j - \sum_{j=n_1+1}^{n_2} A_j + \sum_{j=1}^{n_1} F_j - \sum_{j=n_1+1}^{n_2} F_j S_j + \sum_{j=1}^{n_2} E_j R_j - \sum_{j=n_1+1}^{n_2} E_j - \sum_{j=1}^{n_2} B_j > 0$$

$$|\arg x| < \frac{1}{2} \Omega \pi, |\arg y| < \frac{1}{2} \Lambda \pi$$

The behavior of the $\overline{H}$-function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x, y] = 0(|x|^p, |y|^p), \max \{|x|, |y|\} \to 0$$

Where

$$\alpha = \min_{0 \leq j < m_2} \left[ \text{Re} \left( \frac{d_j}{\delta_j} \right) \right], \quad \beta = \min_{0 \leq j < m_2} \left[ \text{Re} \left( \frac{f_j}{F_j} \right) \right]$$

For large value of $|z|$, the $\overline{H}$-function of two variables reduces to $H$-function of two variables due to [13].

If we set $n_1 = p_1 = q_1 = 0$, the $\overline{H}$-function of two variables breaks up into a product of two $\overline{H}$-function of one variable namely

$$\overline{H}_{0,0}^{m_2,0,0} = \overline{H}_{0,0}^{m_2,0,0} \left[ x \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \right]$$

$$= \overline{H}_{0,0}^{m_2,0,0} \left[ x \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \right]$$

If $\lambda > 0$, we then obtain

$$\lambda^2 \overline{H}_{0,n_1}^{m_2,m_1,0} = \lambda^2 \overline{H}_{0,n_1}^{m_2,m_1,0} \left[ x \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \right]$$

$$= \lambda^2 \overline{H}_{0,n_1}^{m_2,m_1,0} \left[ x \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \left( e^{-j} e^{j} \right)^{-1} \right]$$
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III. Main Results

\[
\begin{align*}
&= \overline{H}_{m_1, n_1}^{0, a_{1}, m_2, n_2, a_{1}} \\
&= \overline{H}_{m_3, n_3}^{0, a_{1}, m_4, n_4, a_{1}} \\
&= \overline{H}_{m_5, n_5}^{0, a_{1}, m_6, n_6, a_{1}} \\
&\text{where } Re \left[ \min \left( \frac{b_i}{\beta_j} \right) \right] > 0; i = 1, 2, ..., m_1; j = 1, 2, ..., m_2; k = 1, 2, ..., m_3 \text{ and }
\lambda_1, \lambda_2, \lambda_3 > 0; | \arg a | < \frac{\pi \lambda_1}{2}, | \arg b | < \frac{\pi \lambda_2}{2}, | \arg c | < \frac{\pi \lambda_3}{2}, \text{where}
\lambda_1 = \sum_{m_1=1}^{n_1} \beta_j - \sum_{m_2=1}^{n_2} B_j \beta_j + \sum_{m_3=1}^{n_3} A_j \alpha_j - \sum_{n_1=1}^{p_1} \alpha_j
\lambda_2 = \sum_{m_1=1}^{n_1} \delta_j - \sum_{m_2=1}^{n_2} D_j \delta_j + \sum_{m_3=1}^{n_3} C_j \gamma_j - \sum_{n_1=1}^{p_1} \gamma_j
\lambda_3 = \sum_{m_1=1}^{n_1} E_j \phi_j - \sum_{m_2=1}^{n_2} \theta_j - \sum_{n_1=1}^{p_1} \phi_j

\text{Proof: On substituting the value of } \overline{H}_{m_1, n_1}^{m_2, n_2, a_{1}} \text{ in terms of Mellin-Barnes integral ([16],p.171)in the integrand of (2.1) and changing the order of integration, the integral transforms into}
\end{align*}
\]

The change of order of integration is readily justified by de la Vallée Poussin’s theorem ([2],p.504)in view of the conditions stated earlier.

On evaluating the $x$-integral by means of ([18],p.1143), it gives us

\[
\begin{align*}
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \phi_{\delta}(\xi) \phi(\eta) \prod_{n_1=1}^{m_1} \Gamma(1 - \phi_j + e_j + (\xi + \eta) \phi_j) \prod_{n_1=1}^{m_1} \Gamma(\theta_j - f_j + (\xi + \eta) \theta_j) d\xi d\eta
\end{align*}
\]

On using (1.7), we arrive at the result (2.1).
IV. Particular cases

(i) If we set \( n_3 = q_3 = 2, \rho_3 = 1, \rho_1 = \rho_2 = 1, \rho_1 = 2 - k + \rho, e_1 = -\rho - m - \frac{1}{2} \)

\[ e_2 = m - \rho - \frac{1}{2}, K_j = L_j = \begin{cases} 1, & j = 1, \end{cases} \]

then on using the identity

\[
H_{2,0} \left[ x \left( \frac{1 - k + \rho, 1}{2^m + m + \rho + 1} \right) \left( \frac{1 - m - \rho}{2^m + m + \rho + 1} \right) \right] = e^{-\frac{1}{2}x^2} W_{k,m}(x) \quad (3.1)
\]

We find that

\[
a^\rho x^2 \int_0^\infty e^{-\frac{1}{2}x^2} W_{k,m}(ax)H_{\rho_1,\rho_2}\left[ \frac{b_{ij}a_{ij}A_{ij}}{c_{ij}A_{ij}} \right] H_{\rho_2,\rho_2} \left[ cx \left( \frac{d_{ij}A_{ij}}{e_{ij}A_{ij}} \right) \right] dx
\]

\[= \frac{\pi^{\rho - m + 3}}{2} \left( \frac{b}{\beta} + \frac{d}{\delta} \right) > 0 \text{ for } i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; \alpha_1, \alpha_2 > 0, \Re(a) > 0, | \arg b | < \frac{1}{2} \pi \lambda_1, | \arg c | < \frac{1}{2} \pi \lambda_2
\]

For \( k = 0, m = \frac{1}{2} \), (3.2) gives Laplace transform of the product of two \( H \)-functions:

\[
a^\rho x^2 \int_0^\infty x^\rho H_{\rho_1,\rho_2}\left[ \frac{b_{ij}a_{ij}A_{ij}}{c_{ij}A_{ij}} \right] H_{\rho_2,\rho_2} \left[ cx \left( \frac{d_{ij}A_{ij}}{e_{ij}A_{ij}} \right) \right] dx
\]

\[= \frac{\pi^{\rho + 1}}{2} \left( \frac{b}{\beta} + \frac{d}{\delta} \right) > 0 \text{ for } i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; \alpha_1, \alpha_2 > 0, \Re(a) > 0, | \arg b | < \frac{1}{2} \pi \lambda_1, | \arg c | < \frac{1}{2} \pi \lambda_2
\]

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