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On Normal and Unitary Polynomial Matrices

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ABSTRACT: *The properties of polynomial hermitian, polynomial normal and polynomial unitary matrices are discussed. A characterization for polynomial normal matrix is obtained.*

Keywords: *Hermitian, normal, unitary matrices, polynomial hermitian, polynomial normal, polynomial unitary matrices.*

I. INTRODUCTION

In matrix theory, we come across some special types of matrices and two among these are normal matrix and unitary matrix. The normal matrix plays an important role in the spectral theory of rectangular matrices and in the theory of generalized inverses. In 1918, the concept of normal matrix with entries from the complex field was introduced by O. Toeplitz who gave a necessary and sufficient condition for a complex matrix to be normal. Unitary matrices have significant importance in quantum mechanics because they preserve norms, and thus, probability amplitudes.

In this paper we have introduced polynomial normal matrix and polynomial unitary matrix. Some properties about these matrices are discussed.

Definition 1.1[1]

A matrix $A \in M_n$ is said to be normal if $A^* A = A A^*$, where A^* is the complex conjugate transpose of A.

Theorem 1.2[1]

If A and B are normal and AB = BA then AB is normal.

Definition 1.3[2]

Let $A \in C_{n\,x\,n}$. The matrix B is said to be unitarily equivalent to A if there exists an unitary matrix U such that $B = U^* AU$.

Theorem 1.4 [3]

If $A \in M_n$ is normal if and only if every matrix unitarily equivalent to A is normal.

Theorem 1.5 [2]

A is normal if and only if it is unitarily similar to a diagonal matrix.

Definition 1.6[2]

A complex matrix $U \in M_n$ is unitary if $U^* U = I$.

Theorem 1.7 [2]

An n x n complex matrix A is unitary if and only if row (or column) vectors form an orthonormal set in C^n.

Theorem 1.8[2]

U is unitary if and only if U^* is unitary.

Theorem 1.9[2]

Determinant value of unitary matrix is unity.

Theorem 1.10[2]

If $U \in M_n$ is unitary then it is diagonalizable.

Definition 1.11[4]

A Polynomial matrix is a matrix whose elements are polynomials.

Example 1.12

Let $A(x) = \begin{pmatrix} 1+x^2 & x+x^2 \\ 1+x & 2+x+x^2 \end{pmatrix} = A_2\lambda^2 + A_1\lambda + A_0$ be a polynomial matrix.

II. Polynomial Normal Matrices

Definition 2.1

A polynomial normal matrix is a polynomial matrix whose coefficient matrices are normal matrices.

Example 2.2

$$A(\lambda) = \begin{pmatrix} i\lambda^2 + i\lambda + 2i & i\lambda^2 + 2i\lambda + i \\ i\lambda^2 + 2i\lambda + i & 2i\lambda^2 + 3i\lambda + i \end{pmatrix}$$

$$= \begin{pmatrix} i & i \\ i & 2i \end{pmatrix}\lambda^2 + \begin{pmatrix} i & 2i \\ 2i & 3i \end{pmatrix}\lambda + \begin{pmatrix} 2i & i \\ i & i \end{pmatrix}$$

$= A_2\lambda^2 + A_1\lambda + A_0$, where $A_0,\ A_1$ *and* A_2 are normal matrices.

Some results on Polynomial normal matrices:

Theorem 2.4

If $A(\lambda)$ and $B(\lambda)$ are polynomial normal matrices and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$ then $A(\lambda)B(\lambda)$ is a polynomial normal matrix.

Proof

Let $A(\lambda) = A_0 + A_1\lambda + \ldots\ldots\ldots\ldots + A_n\lambda^n$ and $B(\lambda) = B_0 + B_1\lambda + \ldots\ldots\ldots\ldots + B_n\lambda^n$ be polynomial normal matrices, $A_0, A_1, A_2, \ldots\ldots, A_n$ *and* $B_0, B_1, B_2, \ldots\ldots, B_n$ are normal matrices. And also given $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$.

$$A(\lambda)B(\lambda) = A_0B_O + (A_0B_1 + A_1B_0)\lambda + \ldots\ldots\ldots + \left(A_0B_n + A_1B_{n-1} + \ldots\ldots\ldots\ldots + A_nB_0\right)\lambda^n$$

$$B(\lambda)A(\lambda) = B_0A_O + (B_0A_1 + B_1A_0)\lambda + \ldots\ldots\ldots + \left(B_0A_n + B_1A_{n-1} + \ldots\ldots\ldots\ldots + B_nA_0\right)\lambda^n.$$

Here each coefficients of λ and constants terms are equal.

i.e. $A_0B_O = B_0A_O$

$A_0B_1 + A_1B_0 = B_0A_1 + B_1A_0 \quad \Rightarrow \quad A_0B_1 = B_0A_1$ and $A_1B_0 = B_1A_0$ ……….

$A_0B_n + A_1B_{n-1} + \ldots\ldots\ldots\ldots + A_nB_0 = B_0A_n + B_1A_{n-1} + \ldots\ldots\ldots\ldots + B_nA_0$

$\Rightarrow \quad A_nB_O = B_0A_n,\ A_1B_{n-1} = B_1A_{n-1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, A_0B_n = B_nA_O.$

Now we prove $A(\lambda)B(\lambda)$ is normal.

$$\begin{aligned}
[A(\lambda)B(\lambda)][A(\lambda)B(\lambda)]^* &= A(\lambda)B(\lambda)[A(\lambda)]^*[B(\lambda)]^* \\
&= A(\lambda)\,[A(\lambda)]^* B(\lambda)[B(\lambda)]^* \\
&= [A(\lambda)]^* A(\lambda)[B(\lambda)]^* B(\lambda) \\
&= [A(\lambda)]^*[B(\lambda)]^* A(\lambda)B(\lambda) \\
&= [A(\lambda)B(\lambda)]^*[A(\lambda)B(\lambda)]
\end{aligned}$$

Hence $A(\lambda)B(\lambda)$ is normal.

Example 2.5

Let $A(\lambda) = \begin{pmatrix} i\lambda^2 + 2i\lambda & -2i\lambda^2 + i \\ 2i\lambda^2 - i & i\lambda^2 + 2i\lambda \end{pmatrix} = \begin{pmatrix} i & -2i \\ 2i & i \end{pmatrix}\lambda^2 + \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}\lambda + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$= A_2\lambda^2 + A_1\lambda + A_0$ be polynomial normal matrix, where A_0, A_1 *and* A_2 are normal matrices and let $B(\lambda) = \begin{pmatrix} 2i\lambda + i & 2i\lambda^2 + i \\ -2i\lambda^2 - i & 2i\lambda + i \end{pmatrix}$

$$= \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}\lambda + \begin{pmatrix} i & i \\ -i & i \end{pmatrix}$$

$= B_2\lambda^2 + B_1\lambda + B_0$ be polynomial normal matrix, where B_0, B_1 *and* B_2 are normal matrices and

$$A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = \begin{pmatrix} -4\lambda^4 - 2\lambda^3 - 5\lambda^2 - 2\lambda + 1 & -2\lambda^4 + \lambda^2 - 4\lambda - 1 \\ 2\lambda^4 - \lambda^2 + 4\lambda + 1 & -4\lambda^4 - 2\lambda^3 - 5\lambda^2 - 2\lambda + 1 \end{pmatrix}.$$

Now we have to prove $A(\lambda)B(\lambda)$ is normal.

That is, $A(\lambda)B(\lambda)][A(\lambda)B(\lambda)]^* = [A(\lambda)B(\lambda)]^*[A(\lambda)B(\lambda)]$

$$= \begin{pmatrix} 20\lambda^8 + 16\lambda^7 + 40\lambda^6 + 52\lambda^5 + 30\lambda^4 + 8\lambda^3 + 8\lambda^2 + 4\lambda + 2 & 0 \\ 0 & 20\lambda^8 + 16\lambda^7 + 40\lambda^6 + 52\lambda^5 + 30\lambda^4 + 8\lambda^3 + 8\lambda^2 + 4\lambda + 2 \end{pmatrix}$$

Hence $A(\lambda)B(\lambda)$ is normal.

Theorem 2.6

If $A(\lambda)$ is a polynomial normal matrix if and only if every polynomial matrix unitarily equivalent to $A(\lambda)$ is polynomial normal matrix.

Proof

Suppose $A(\lambda)$ is polynomial normal matrix and $B(\lambda) = U(\lambda)^* A(\lambda)U(\lambda)$, where $U(\lambda)$ is polynomial unitary matrix. Now we show that $B(\lambda)$ is polynomial normal matrix.

$$\begin{aligned} B(\lambda)^* B(\lambda) &= (U(\lambda)^* A(\lambda)^* U(\lambda))\,(U(\lambda)^* A(\lambda)U(\lambda)) \\ &= U(\lambda)^* A(\lambda)^* A(\lambda)U(\lambda) \\ &= U(\lambda)^* A(\lambda)A(\lambda)^* U(\lambda) \\ &= (U(\lambda)^* A(\lambda)U(\lambda))\,(U(\lambda)^* A(\lambda)^* U(\lambda)) \\ &= B(\lambda)B(\lambda)^* \end{aligned}$$

Hence $B(\lambda)$ is polynomial normal matrix.

Conversely, assume $B(\lambda)$ is polynomial normal matrix. To prove $A(\lambda)$ is polynomial normal matrix. To prove A (λ) is polynomial normal matrix

$B(\lambda) = (U(\lambda)^* A(\lambda)U(\lambda))$ is normal

$$\Rightarrow (U(\lambda)^* A(\lambda)U(\lambda))\,(U(\lambda)^* A(\lambda)^* U(\lambda)) = (U(\lambda)^* A(\lambda)^* U(\lambda))\,(U(\lambda)^* A(\lambda)U(\lambda))$$

$$\Rightarrow U(\lambda)^* A(\lambda)(U(\lambda)\,U(\lambda)^*)A(\lambda)U(\lambda) = U(\lambda)^* A(\lambda)^*(U(\lambda)\,U(\lambda)^*)A(\lambda)U(\lambda)$$

$$\Rightarrow U(\lambda)^* A(\lambda)A(\lambda)^*\ (U(\lambda) = U(\lambda)^* A(\lambda)^* A(\lambda)U(\lambda)$$

Pre multiply by $U(\lambda)$ and post multiply by $U(\lambda)^*$, we get

$$A(\lambda)A(\lambda)^* = A(\lambda)^* A(\lambda)$$

Hence $A(\lambda)$ is a polynomial normal matrix.

Result 2.7

Let $A(\lambda)=A_0+A_1\lambda+.....+A_m\lambda^m$ and $B(\lambda)=B_0+B_1\lambda+.....+B_m\lambda^m$ be polynomial matrix, where A_i's, B_i's $\in M_n(C)$, i = 0,1,2,……….m are normal matrices and $A_0B_0 = A_1B_1 = ……A_mB_m = 0$. Then $A_0^*B_0 = A_1^*B_1 = ……A_m^*B_m = 0$.

Proof

Let $A(\lambda)=A_0A_1\lambda+A_2\lambda^2+.....+A_m\lambda^m$ is polynomial normal matrix, where A_0, A_1....... A_m are normal matrices. And $B(\lambda)=B_0B_1\lambda+B_2\lambda^2.....+B_m\lambda^m$ is a polynomial matrix and also given $A_0B_0 = A_1B_1 = ……A_mB_m = 0$. We know that "If A is normal and AB =0 then A^*B = 0."

From this result $A_0^*B_0 = A_1^*B_1 = ……A_m^*B_m = 0$.

III. Polynomial Unitary Matrices

Definition 3.1

A Polynomial unitary matrix is a Polynomial matrix whose coefficient matrices are unitary matrices.

Definition 3.2

Let $A(\lambda), B(\lambda) \in M_n[C(\lambda)]$ where $A_i's, B_i's \in M_n[C(\lambda)]$ i= 1, 2, ………, m. The polynomial matrix $B(\lambda)$ is said to be unitarily equivalent to $A(\lambda)$ if there exists a polynomial unitary matrix $U(\lambda)$ such that $B(\lambda) = U(\lambda)^* A(\lambda) U(\lambda)$.

Theorem 3.3

If $U(\lambda)$ is a Polynomial unitary matrix if and only if $U(\lambda)^*$ is Polynomial unitary matrix.

Proof

Let $U(\lambda)=A_0+A_1\lambda+A_2\lambda^2+.....+A_n\lambda^n$ be polynomial matrix. Here A_i's are unitary matrices. (ie) $A_0A^*_0=I, A_1A^*_1=I,...........A_nA_n^*=I.$

To prove $U(\lambda)^*$ is polynomial unitary matrix

From (1), $U(\lambda)^* = A_0^* + A^*_1\lambda+.....+A^*_n\lambda^n$ We know that A_i's are unitary matrices

Hence $U(\lambda)^*$ is Polynomial unitary matrix . Similarly we can prove the converse.

Remark: 3.4

Since all the coefficient matrices of a polynomial unitary matrix is unitary their determinant value is one.

Theorem:3.5

If $U(\lambda)=A_0+A_1\lambda+A_2\lambda^2+.....+A_n\lambda^n$ is polynomial unitary matrix, where $A_i \in M_n[F]$ and $A_0, A_1, A_2,....A_n$ are unitary matrices then A_i's are diagonalizable.

Proof

Given $U(\lambda)=A_0+A_1\lambda+.....+A_n\lambda^n$ is a polynomial unitary matrix, where A_0, A_1....... A_n are unitary matrices. By the result 1.17, we have unitary matrices are diagonalizable.

Therefore A_0, A_1....... A_n are diagonalizable.

Result: 3.6

If $A(\lambda)$ is a polynomial unitary matrix then the absolute value of all the eigen values of the coefficient matrices are unity.

IV. Polynomial Normal And Polynomial Unitary Matrices

Theorem 4.1

Polynomial unitary matrices are polynomial normal matrices.

Proof

We know that unitary matrices are normal. In polynomial unitary matrix all the coefficient matrices are unitary. Hence by the definition 2.1, we get polynomial unitary matrices are polynomial normal matrices.

Definition 4.2

A polynomial hermitian matrix is a polynominal matrix whose coefficients are hermitian matrices.
Theorem 4.3

Polynomial hermitian matrix is polynomial normal matrix if the coefficient matrices satisfy $AA^* = A^2 = A^*A$.

Proof

We know that hermitian matrix with $AA^* = A^2 = A^*A$ is normal. In polynomial hermitian matrix we have all the coefficient matrices are hermitian. By our hypothesis we have coefficient matrices satisfy the condition $AA^* = A^2 = A^*A$. Now we get all the coefficient matrices are normal.
Hence the theorem.

V. Conclusion

In this paper some of the properties of polynomial hermitian, polynomial normal and polynomial unitary matrices are derived. Similarly we can extend all the properties of hermitian, normal and unitary matrices to polynomial hermitian, normal and unitary matrices.

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