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On The Fractional Euler Top System with Two Parameters

Mihai Ivan * and Gheorghe Ivan **
West University of Timişoara
Department of Mathematics. Seminarul de Geometrie şi Topologie
4, B-dul V. Pârvan, 300223, Timişoara, Romania
Corresponding Author: Mihai Ivan

We present the dynamical behavior of a family of fractional differential systems associated to Euler top system with two parameters. The fractional stability of equilibrium states for the perturbed fractional Euler top system with two linear controls is studied. Finally, the numerical integration and numerical simulation of them are discussed.
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I. INTRODUCTION

The Hamilton-Poisson systems appear naturally in many areas of physical science and engineering, robotics, spatial dynamics and secure communications [9, 3]. A special class of Hamilton-Poisson systems is formed by a family of differential equations on $\mathbf{R}^3$ which depends by a triple of real parameters $(\alpha_1, \alpha_2, \alpha_3)$ and two parameters a and b, called the (general) Euler top system with two parameters. This family contains various integrable systems, for instance: a particular case of Rikitake system, a special case of Rabinovich system and many others. The Rikitake system serves as model for the reversals of polarity of the Earth's magnetic field [14]. The Rabinovich system models the dynamics of three resonantly coupled waves, parametrically excited [12].

The fractional calculus has been found to be an important tool in various fields, such as mathematics, physics, engineering, chemistry, biology, economics, chaotic dynamics and other complex dynamical systems [1, 5, 7, 11, 13]. A class of fractional dynamical systems is formed by a family of fractional differential equations on $\mathbf{R}^3$ associated to general Euler top system with two parameters, called the (general) fractional Euler top system with two parameters.

This paper is structured as follows. In Section 2, the general Euler top system with two parameters is realized as a Hamilton-Poisson system. In Section 3 we introduce the fractional Euler top system with two linear controls (3.1). This section is devoted to studying of the fractional stability of equilibrium states for the fractional system (3.1). In Section 4, the numerical integration and numerical simulation for the perturbed fractional Euler top system associated to dynamics (3.1) is discussed.

II. GENERAL EULER TOP SYSTEM WITH TWO PARAMETERS

For details on Hamiltonian dynamics, see e.g. [9].

We consider the following differential system of Euler type on $\mathbf{R}^3$:

$$\begin{cases} \dot{x}^1(t) = \alpha_1 x^2(t) x^3(t) + a x^2(t) \\ \dot{x}^2(t) = \alpha_2 x^1(t) x^3(t) + b x^1(t), \\ \dot{x}^3(t) = \alpha_3 x^1(t) x^2(t) \end{cases} \tag{2.1}$$

where $\dot{x}^i(t) = dx^i(t)/dt$, $\alpha_i, a, b \in \mathbf{R}$ for $i = \overline{1,3}$ such that $\alpha_1\alpha_2\alpha_3 \neq 0$ and t is the time.

If $ab \neq 0$, (2.1) is called the *(general) Euler top system with two linear controls.*
If $a = b = 0$, (2.1) becomes the *(general) Euler top system* [6], given by:

$$\dot{x}^1(t) = \alpha_1 x^2(t)x^3(t); \quad \dot{x}^2(t) = \alpha_2 x^1(t)x^3(t); \quad \dot{x}^3(t) = \alpha_3 x^1(t)x^2(t), \tag{2.2}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that $\alpha_1\alpha_2\alpha_3 \neq 0$.

We will denote the vector of parameters $\alpha_i, i = \overline{1,3}$ by $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Remark 2.1. In [6] is showed that the general Euler top system may be realized as a Hamilton-Poisson system in an infinite number of different ways. □

Example 2.1. (*i*) Let $a_1 = \dfrac{1}{I_1}, a_2 = \dfrac{1}{I_2}, a_3 = \dfrac{1}{I_3}$ such that $I_1 > I_2 > I_3 > 0$.

If in (2.1) we take $\alpha = (a_3 - a_2, a_1 - a_3, a_2 - a_1)$, $a, b \in \mathbf{R}^*$, then:

$$\dot{x}^1 = (a_3 - a_2)x^2x^3 + ax^2; \quad \dot{x}^2 = (a_1 - a_3)x^1x^3 + bx^1; \quad \dot{x}^3 = (a_2 - a_1)x^1x^2. \tag{2.3}$$

The dynamics (2.3) is called the *rigid body equations with two linear controls*, where $x = (x^1, x^2, x^3)$ represents the angular velocity vector of rigid body, I_1, I_2, I_3 the components of its inertia tensor and $a, b \in \mathbf{R}^*$ are parameters.

If in (2.3) we take $a = b = 0$, then we obtain the *Euler equations of the free rigid body on orthogonal group* $SO(3)$ [9].

(*ii*) For $\alpha = (k, 1, -1)$ and $a = -b = \beta$, the system (2.1) becomes:

$$\dot{x}^1 = kx^2x^3 + \beta x^2; \quad \dot{x}^2 = x^1x^3 - \beta x^1; \quad \dot{x}^3 = -x^1x^2, \tag{2.4}$$

where $k, \beta \in \mathbf{R}$ with $k < 0$ are parameters. This system is called the *Rikitake-Hamilton system with one quadratic control* [14, 8].

(*iii*) For $\alpha = (1, -1, 1)$ and $a = b = \beta \in \mathbf{R}$, the system (2.1) becomes:

$$\dot{x}^1 = x^2x^3 + \beta x^2; \quad \dot{x}^2 = -x^1x^3 + \beta x^1; \quad \dot{x}^3 = x^1x^2. \tag{2.5}$$

Note that the system (2.5) is the general Rabinovich studied in [12] in the case $\nu_1 = \nu_2 = \nu_3 = 0$. It is called the *Rabinovich system with two linear controls*. For $\beta = 0$, the dynamics (2.5) is the *Rabinovich system* [2, 15]. □

Remark 2.2. Among the studied topics related to the systems given in Example 2.1 we recall the construction of Hamilton-Poisson realizations and nonlinear stability problem for the systems (2.3) with $k = 0$ [9], (2.4) [8] and (2.5) with $\beta = 0$ [2, 15]. □

Proposition 2.1. *Let* $a, b \in \mathbf{R}, a \neq 0$. *A Hamilton-Poisson realization of the Euler top system* (2.1) *is* $(\mathbf{R}^3, P^{\alpha}_{a,b}, H^{\alpha}_{a,b})$ *with the Casimir* $C^{\alpha}_{a,b} \in C^{\infty}(\mathbf{R}^3, \mathbf{R})$, *where*

$$P^{\alpha}_{a,b} = \begin{pmatrix} 0 & \dfrac{b\alpha_1 - a\alpha_2}{a\alpha_3}x^3 & -x^2 \\ \dfrac{a\alpha_2 - b\alpha_1}{a\alpha_3}x^3 & 0 & -\dfrac{b}{a}x^1 \\ x^2 & \dfrac{b}{a}x^1 & 0 \end{pmatrix}, \tag{2.6}$$

$$H^{\alpha}_{a,b} = \frac{1}{2}\left[(\alpha_3 - b)(x^1)^2 + a(x^2)^2 + \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3}(x^3)^2\right] - ax^3, \tag{2.7}$$

$$C^{\alpha}_{a,b} = \frac{1}{2}\left[b\alpha_3(x^1)^2 - a\alpha_3(x^2)^2 + (a\alpha_2 - b\alpha_1)(x^3)^2\right] \tag{2.8}$$

Proof. We have $\frac{\partial H_{a,b}^{\alpha}}{\partial x^1} = (\alpha_3 - b)x^1$, $\frac{\partial H_{a,b}^{\alpha}}{\partial x^2} = ax^2$, $\frac{\partial H_{a,b}^{\alpha}}{\partial x^3} = \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3} x^3 - a,$

$\frac{\partial C_{a,b}^{\alpha}}{\partial x^1} = b\alpha_3 x^1$, $\frac{\partial C_{a,b}^{\alpha}}{\partial x^2} = -a\alpha_3 x^2$, $\frac{\partial C_{a,b}^{\alpha}}{\partial x^3} = (a\alpha_2 - b\alpha_1)x^3.$

Then $P_{a,b}^{\alpha} \cdot \nabla H_{a,b}^{\alpha} =$

$$P_{a,b}^{\alpha} \cdot \nabla H_{a,b}^{\alpha} = \begin{pmatrix} 0 & \frac{b\alpha_1 - a\alpha_2}{a\alpha_3} x^3 & -x^2 \\ \frac{a\alpha_2 - b\alpha_1}{a\alpha_3} x^3 & 0 & -\frac{b}{a} x^1 \\ x^2 & \frac{b}{a} x^1 & 0 \end{pmatrix} \cdot \begin{pmatrix} (\alpha_3 - b)x^1 \\ ax^2 \\ \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3} x^3 - a \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_1 x^2 x^3 + ax^2 \\ \alpha_2 x^1 x^3 + bx^1 \\ \alpha_3 x^1 x^2 \end{pmatrix} = \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{pmatrix}.$$

Hence $\dot{X}(t) = P_{a,b}^{\alpha} \cdot \nabla H_{a,b}^{\alpha}$, where

$\dot{X}(t) = \left(\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)\right)^T$, $\nabla H_{a,b}^{\alpha} = \left(\frac{\partial H_{a,b}^{\alpha}}{\partial x^1}, \frac{\partial H_{a,b}^{\alpha}}{\partial x^2}, \frac{\partial H_{a,b}^{\alpha}}{\partial x^3}\right)^T$ and (2.1) is an Hamilton-Poisson system. Also, $C_{a,b}^{\alpha}$ is a Casimir, since $P_{a,b}^{\alpha} \cdot \nabla C_{a,b}^{\alpha} = 0.$ □

Corollary 2.1. (*i*) *A Hamilton-Poisson realization of the Rikitake-Hamilton system* (2.4) *is* $(\mathbf{R}^3, P^k, H_{\beta}^k)$ *with the Casimir* $C^k \in C^{\infty}(\mathbf{R}^3, \mathbf{R})$, *where*

$$P^k = \begin{pmatrix} 0 & (k+1)x^3 & -x^2 \\ -(k+1)x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix},$$

$$H_{\beta}^k = \frac{1}{2}\left[(\beta - 1)(x^1)^2 + \beta(x^2)^2 + \left(\beta(k+1) - k\right)(x^3)^2\right] - \beta x^3,$$

$$C^k = \frac{1}{2}\left[(x^1)^2 - (x^2)^2 + (k+1)(x^3)^2\right].$$

(*ii*) *A Hamilton-Poisson realization of the Rabinovich system* (2.4) *is* $(\mathbf{R}^3, P, H_{\beta})$ *with the Casimir* $C \in C^{\infty}(\mathbf{R}^3, \mathbf{R})$, *where*

$$P = \begin{pmatrix} 0 & 2x^3 & -x^2 \\ -2x^3 & 0 & -x^1 \\ x^2 & x^1 & 0 \end{pmatrix},$$

$$H_{\beta} = \frac{1}{2}\left[(1-\beta)(x^1)^2 + \beta(x^2)^2 + (2\beta - 1)(x^3)^2\right] - \beta x^3, \qquad C = \frac{1}{2}\left[(x^1)^2 - (x^2)^2 - 2(x^3)^2\right].$$

Proof. The above assertions follows from Proposition 2.1 by replacement of parameters $\alpha_1, \alpha_2, \alpha_3, a$ and b with the corresponding values. □

Proposition 2.2. *The functions* $H^{\alpha}_{a,b}, C^{\alpha}_{a,b} \in C^{\infty}(\mathbf{R}^3, \mathbf{R}),$ *g iven by* (2.7) *and* (2.8) *are constants of the motion (first integrals) for the dynamics* (2.1).
Proof. Indeed,

$$dH^{\alpha}_{a,b}/dt = (\alpha_3 - b)x^1\dot{x}^1 + ax^2\dot{x}^2 + \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3}x^3\dot{x}^3 - a\dot{x}^3 = (\alpha_3 - b)x^1(\alpha_1 x^2 x^3 + ax^2) +$$

$$+ ax^2(\alpha_2 x^1 x^3 + bx^1) + \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3}x^3(\alpha_3 x^1 x^2) - a(\alpha_3 x^1 x^2) = 0.$$

Similarly, we have $dC^{\alpha}_{a,b}/dt = 0.$ □

Remark 2.3. Since $H^{\alpha}_{a,b}$ and $C^{\alpha}_{a,b}$ are first integrals, it follows that *the trajectories of motion of the dynamics* (2.1) *are intersections of the surfaces:*

$$(\alpha_3 - b)(x^1)^2 + a(x^2)^2 + \frac{b\alpha_1 - a\alpha_2 - \alpha_1\alpha_3}{\alpha_3}(x^3)^2 - 2ax^3 = \text{constant} \quad \textit{and}$$

$$b\alpha_3(x^1)^2 - a\alpha_3(x^2)^2 + (a\alpha_2 - b\alpha_1)(x^3)^2 = \text{constant.} \ \square$$

III. FRACTIONAL EULER TOP SYSTEM WITH TWO LINEAR CONTROLS

Let $f \in C^{\infty}(\mathbf{R})$ and $q \in \mathbf{R}, q > 0.$ The $q-$order Caputo differential operator [5], is described by $D^q_t f(t) = I^{m-q} f^{(m)}(t), q > 0,$ where $f^{(m)}(t)$ represents the $m-$ order derivative of the function f, $m \in \mathbf{N}^*$ is an integer such that $m - 1 \le q \le m$ and I^q is the $q-$order Riemann-Liouville integral operator [13], which is expressed as follows:

$$I^q f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds, \qquad q > 0,$$

where Γ is the Euler Gamma function. If $q = 1,$ then $D^1_t f(t) = df/dt.$
In this paper we suppose that $q \in (0,1].$
The *fractional Euler top system with two linear controls associated to dynamics* (2.1) is defined by the following set of fractional differential equations:

$$\begin{cases} D^q_t x^1(t) = \alpha_1 x^2(t)x^3(t) + ax^2(t) \\ D^q_t x^2(t) = \alpha_2 x^1(t)x^3(t) + bx^1(t), \qquad q \in (0,1) \\ D^q_t x^3(t) = \alpha_3 x^1(t)x^2(t) \end{cases} \qquad (3.1)$$

where $\alpha_i, a, b \in \mathbf{R}$ for $i = \overline{1,3}.$

Example 3.1. (i) If in (3.1) we take $\alpha = (k,1,-1)$ with $k < 0$ and $a = -b = \beta \in \mathbf{R},$ then then one obtains the *fractional Rikitake-Hamilton system*, given by:

$$D^q_t x^1 = kx^2x^3 + \beta x^2; \quad D^q_t x^2 = x^1x^3 - \beta x^1; \quad D^q_t x^3 = -x^1x^2. \qquad (3.2)$$

(ii) For $\alpha = (1,-1,1)$ and $a = b = \beta \in \mathbf{R},$ the system (3.1) becomes:

$$D^q_t x^1 = x^2x^3 + \beta x^2; \quad D^q_t x^2 = -x^1x^3 + \beta x^1; \quad D^q_t x^3 = x^1x^2. \qquad (3.3)$$

The system (3.3) is called the *fractional Rabinovich system with two linear controls*.
For $\beta = 0,$ it is the *fractional Rabinovich system*. □

Proposition 3.1. *The equilibrium states of the fractional Euler top system* (3.1) *are given as the union of the following three families*:

$$E_1 := \left\{ e_1^m = (m,0,-\frac{b}{\alpha_2}) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\}, \quad E_2 := \left\{ e_2^m = (0,m,-\frac{a}{\alpha_1}) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\},$$
$$E_3 := \left\{ e_3^m = (0,0,m) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\}.$$

Proof. The equilibrium states are solutions of the equations $f_i(x) = 0, i = \overline{1,3},$ where $f_1(x) = \alpha_1 x^2 x^3 + a x^2$, $f_2(x) = \alpha_2 x^1 x^3 + b x^1$ and $f_3(x) = \alpha_3 x^1 x^2$. □

Let us we present the study of fractional stability of equilibrium states for the system (3.1). Finally, we will discuss how to stabilize the unstable equilibrium states of system (3.1)
via fractional order derivative. For this study we apply the Matignon's test [10].
The *Jacobian matrix associated to system* (3.1) is:

$$J_{a,b}(x) = \begin{pmatrix} 0 & \alpha_1 x^3 + a & \alpha_1 x^2 \\ \alpha_2 x^2 + b & 0 & \alpha_2 x^1 \\ \alpha_3 x^2 & \alpha_3 x^1 & 0 \end{pmatrix}.$$

Proposition 3.2. ([10]) *Let* x_e *be an equilibrium state of system* (3.1) *and* $J_{a,b}(x_e)$ *be the Jacobian matrix* $J_{a,b}(x)$ *evaluated at* x_e.

(i) x_e *is locally asymptotically stable, iff all eigenvalues of the matrix* $J_{a,b}(x_e)$ *satisfy:*

$$|\arg(\lambda(J_{a,b}(x_e)))| > \frac{q\pi}{2}.$$

(ii) x_e *is locally stable, iff either it is asymptotically stable, or the critical eigenvalues of* $J_{a,b}(x_e)$ *which satisfy* $|\arg(\lambda(J_{a,b}(x_e)))| = \frac{q\pi}{2}$ *have geometric multiplicity one.* □

Proposition 3.3. *The equilibrium states* $e_i^m, i = \overline{1,3}$ *are unstable* $(\forall) q \in (0,1)$.

Proof. The characteristic polynomial of the matrix $J_{a,b}(e_1^m) = \begin{pmatrix} 0 & -\frac{b\alpha_1}{\alpha_2} + a & 0 \\ 0 & 0 & m\alpha_2 \\ 0 & m\alpha_3 & 0 \end{pmatrix}$ is

$$p_{J_{a,b}(e_1^m)}(\lambda) = \det(J_{a,b}(e_1^m) - \lambda I) = -\lambda(\lambda^2 - \alpha_2\alpha_3 m^2).$$

The characteristic polynomials of matrices $J_{a,b}(e_2^m)$ and $J_{a,b}(e_3^m)$ are the following:

$$p_{J_{a,b}(e_2^m)}(\lambda) = -\lambda(\lambda^2 - \alpha_1\alpha_3 m^2) \quad \text{and} \quad p_{J_{a,b}(e_3^m)}(\lambda) = -\lambda[\lambda^2 - (\alpha_1 m + a)(\alpha_2 m + b)].$$

The equations $p_{J_{a,b}(e_1^m)}(\lambda) = 0,\ p_{J_{a,b}(e_2^m)}(\lambda) = 0,\ p_{J_{a,b}(e_3^m)}(\lambda) = 0$ have the root $\lambda_1 = 0$.

Since $\arg(\lambda_1) = 0 < \frac{q\pi}{2}$ for all $q \in (0,1)$, by Proposition 3.1 follows that the equilibrium states $e_i^m, i = \overline{1,3}$ are unstable for all $q \in (0,1)$. □

In the case when x_e is a unstable equilibrium state of the fractional system (3.1), we associate to (3.1) a new fractional system, called the *perturbed fractional Euler top system with two linear controls for the equilibrium state* x_e.
If one selects the parameters which then make the eigenvalues of Jacobian matrix

of the perturbed fractional system associated to system (3.1) for the equilibrium state x_e satisfy one of the conditions from Proposition 3.2, then the trajectories of them asymptotically approaches the unstable equilibrium state x_e in the sense that $\lim_{t\to\infty} \| x(t) - x_e \| = 0,$ where $\|\cdot\|$ is the Euclidean norm.
The *perturbed fractional Euler top system with two linear controls for the equilibrium state* $e_1^m = (m, 0, -\frac{b}{\alpha_2})$ is defined by:

$$\begin{cases} D_t^q x^1(t) = \alpha_1 x^2(t) x^3(t) + a x^2(t) + c_{11}(x^1 - m) \\ D_t^q x^2(t) = \alpha_2 x^1(t) x^3(t) + b x^1(t) + c_{12} x^2, \qquad q \in (0,1) \\ D_t^q x^3(t) = \alpha_3 x^1(t) x^2(t) + c_{12}\left(x^3 + \dfrac{b}{\alpha_3} \right) \end{cases} \tag{3.4}$$

where $c_{11}, c_{12} \in \mathbf{R}$ are real constants.
The Jacobian matrix of the perturbed fractional Euler top system (3.4) for e_1^m is

$$J_{a,b}(x, c_{11}, c_{12}) = \begin{pmatrix} c_{11} & \alpha_1 x^3 + a & \alpha_1 x^2 \\ \alpha_2 x^2 + b & c_{12} & \alpha_2 x^1 \\ \alpha_3 x^2 & \alpha_3 x^1 & c_{12} \end{pmatrix}.$$

Proposition 3.4. *Let be the perturbed fractional Euler top system* (3.4).
(i) *Let* $\alpha_2 \alpha_3 < 0$ *and* $m \in \mathbf{R}$.
(1) If $c_{11} < 0, c_{12} < 0,$ *then* e_1^m *is asymptotically stable* $(\forall) q \in (0,1);$
(2) If $c_{11} < 0, c_{12} > 0,$ *then* e_1^m *is asymptotically stable* $(\forall) q \in (0, q_1),$ *where* $q_1 = \frac{2}{\pi} | \arctan \frac{|m| \sqrt{|\alpha_2 \alpha_3|}}{|c_{12}|} |$ *and it is stable for* $q = q_1$.
(ii) Let $\alpha_2 \alpha_3 > 0.$ *If* $c_{11} < 0, c_{12} < 0,$ *then* e_1^m *is asymptotically stable for all* $m \in \left(\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}}, -\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}} \right)$ *and* $q \in (0,1)$.

Proof. The characteristic polynomial of $J_{a,b}(e_1^m, c_{11}, c_{12}) = \begin{pmatrix} c_{11} & -\frac{b\alpha_1}{\alpha_2} + a & 0 \\ 0 & c_{12} & m\alpha_2 \\ 0 & m\alpha_3 & c_{12} \end{pmatrix}$

is $p_1(\lambda) = -(\lambda - c_{11})[(\lambda - c_{12})^2 - \alpha_2 \alpha_3 m^2]$. The roots of the characteristic equation $p_1(\lambda) = 0$ are $\lambda_1 = c_{11}$, $\lambda_{2,3} = c_{12} \pm m\sqrt{|\alpha_2 \alpha_3|}$.
(i) **Case** $\alpha_2 \alpha_3 < 0.$ Then $\lambda_1 = c_{11}$, $\lambda_{2,3} = c_{12} \pm i m \sqrt{|\alpha_2 \alpha_3|}$.
(i.1) We suppose that $c_{11} < 0$ and $c_{12} < 0.$ In this case we have $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0.$ Since $|\arg(\lambda_i)| = \pi > \frac{q\pi}{2}$, $i = \overline{1,3}$ for all $q \in (0,1),$ by Proposition 3.2(i), it implies that e_1^m is locally asymptotically stable for all $m \in \mathbf{R}$.
(i.2) For $c_{11} < 0$ and $c_{12} > 0,$ we have $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) > 0.$ Applying Proposition 3.2(i),

e_1^m is locally asymptotically stable, for $0 < q < q_1$, where $q_1 = \frac{2}{\pi} | \arctan \frac{|m| \sqrt{|\alpha_2 \alpha_3|}}{|c_{12}|} |$.

Hence, the assertion (i.2) holds. If $q = q_1$, then e_1^m is stable. For $q_1 < q < 1$, e_1^m is unstable $(\forall) m \in \mathbf{R}$.

(i.3) Let $c_{11} > 0$ and $c_{12} \in \mathbf{R}$. Since $\lambda_1 > 0$, $J_{a,b}(e_1^m, c_{11}, c_{12})$ has at least a positive eigenvalue and so e_1^m is unstable for all $m \in \mathbf{R}$. Therefore, the assertions (i) hold.

(ii) **Case** $\alpha_2 \alpha_3 > 0$. Then $\lambda_1 = c_{11}$, $\lambda_{2,3} = c_{12} \pm m\sqrt{|\alpha_2 \alpha_3|}$.

(ii.1) Suppose $c_{11} < 0$ and $c_{12} < 0$. Then $\lambda_1 < 0$, $\lambda_2 = c_{12} + m\sqrt{|\alpha_2 \alpha_3|} < 0$ for $m \in \left(-\infty, -\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}} \right)$ and $\lambda_2 = c_{12} - m\sqrt{|\alpha_2 \alpha_3|} < 0$ for $m \in \left(\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}}, +\infty \right)$. Since the eigenvalues are all negative for $m \in \left(\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}}, -\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}} \right)$ it follows that e_1^m is asymptotically stable for all $q \in (0,1)$.

(ii.2) If $c_{11} < 0$ and $c_{12} > 0$, then $\lambda_1 < 0$, $\lambda_2 = c_{12} + m\sqrt{|\alpha_2 \alpha_3|} < 0$ for $m \in \left(-\infty, -\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}} \right)$ and $\lambda_2 = c_{12} - m\sqrt{|\alpha_2 \alpha_3|} < 0$ for $m \in \left(\frac{c_{12}}{\sqrt{|\alpha_2 \alpha_3|}}, +\infty \right)$. Hence, $J_{a,b}(e_1^m, c_{11}, c_{12})$ has at least a positive eigenvalue and so e_1^m is unstable.

(ii.2) Let $c_{11} > 0$ and $c_{12} \in \mathbf{R}$. Since $\lambda_1 > 0$, $J_{a,b}(e_1^m, c_{11}, c_{12})$ has at least a positive eigenvalue and so e_1^m is unstable $(\forall) m \in \mathbf{R}$. Hence, the assertion (ii) holds. □

If in (3.4) we take $\alpha = (k,1,-1)$ with $k < 0$ and $a = -b = \beta \in \mathbf{R}^*$, then one obtains the *perturbed fractional Rikitake-Hamilton system for* $e_1^m = (m, 0, \beta)$.

Corollary 3.1. *Let be the perturbed fractional Rikitake-Hamilton system for* $e_1^m = (m, 0, \beta)$. *If* $c_{11} < 0$ *and* $c_{12} < 0$, *then* e_1^m *is asymptotically stable* $(\forall) m \in \mathbf{R}$ *and* $q \in (0,1)$.

Proof. Since $\alpha_2 \alpha_3 = -1 < 0$, the assertion follows from Proposition 3.4(i). □

The *perturbed fractional Euler top system with two linear controls for the equilibrium state* $e_2^m = (0, m, -\frac{a}{\alpha_1})$ is defined by:

$$\begin{cases} D_t^q x^1(t) = \alpha_1 x^2(t) x^3(t) + a x^2(t) + c_{21} x^1 \\ D_t^q x^2(t) = \alpha_2 x^1(t) x^3(t) + b x^1(t) + c_{22}(x^2 - m), \\ D_t^q x^3(t) = \alpha_3 x^1(t) x^2(t) + c_{21} \left(x^3 + \frac{a}{\alpha_1} \right) \end{cases} \qquad q \in (0,1) \qquad (3.5)$$

where $c_{21}, c_{22} \in \mathbf{R}$ are real constants.

The Jacobian matrix of the perturbed fractional Euler top system (3.5) for e_2^m is

$$J_{a,b}(x,c_{21},c_{22})=\begin{pmatrix} c_{21} & \alpha_1 x^3+a & \alpha_1 x^2 \\ \alpha_2 x^2+b & c_{22} & \alpha_2 x^1 \\ \alpha_3 x^2 & \alpha_3 x^1 & c_{21} \end{pmatrix}.$$

Proposition 3.5. *Let be the perturbed fractional Euler top system* (3.5).

(i) Let $\alpha_1\alpha_3<0$ *and* $m\in\mathbf{R}$.

(1) If $c_{21}<0, c_{22}<0,$ *then* e_2^m *is asymptotically stable* $(\forall)q\in(0,1)$;

(2) If $c_{21}<0, c_{22}>0,$ *then* e_2^m *is asymptotically stable* $(\forall)q\in(0,q_2),$ *where* $q_2=\frac{2}{\pi}\left|\arctan\frac{|m|\sqrt{|\alpha_1\alpha_3|}}{|c_{21}|}\right|$ *and it is stable for* $q=q_2$.

(ii) Let $\alpha_1\alpha_3>0$. *If* $c_{21}<0, c_{22}<0,$ *then* e_2^m *is asymptotically stable for all* $m\in\left(\frac{c_{21}}{\sqrt{|\alpha_1\alpha_3|}},-\frac{c_{21}}{\sqrt{|\alpha_1\alpha_3|}}\right)$ *and* $q\in(0,1)$.

Proof. Let be $p_2(\lambda)$ the characteristic polynomial of matrix $J_{a,b}(e_2^m,c_{21},c_{22})$. The roots of the characteristic roots of $p_2(\lambda)=-(\lambda-c_{22})[(\lambda-c_{21})^2-\alpha_1\alpha_3 m^2]$ are $\lambda_1=c_{22}$, $\lambda_{2,3}=c_{21}\pm m\sqrt{|\alpha_1\alpha_3|}$.

Applying the same reasoning as in the demonstration of Proposition 3.4 it is easy to prove that the the assertions (i) and (ii) hold. □

If in (3.5) we take $\alpha=(1,-1,1)$ and $a=b=\beta\in\mathbf{R},$ then one obtains the *perturbed fractional Rabinovich system for* $e_2^m=(0,m,-\beta)$.

Corollary 3.2. *Let be the perturbed fractional Rabinovich system for* $e_2^m=(0,m,-\beta)$.

If $c_{21}<0$ *and* $c_{22}<0,$ *then* e_2^m *is asymptotically stable* $(\forall)m\in(c_{21},-c_{21})$ *and* $q\in(0,1)$.

Proof. Since $\alpha_1\alpha_3=1>0,$ the assertion follows from Proposition 3.5 (ii). □

The *perturbed fractional Euler top system with two linear controls for the equilibrium state* $e_3^m=(0,0,m)$ is defined by:

$$\begin{cases} D_t^q x^1(t)=\alpha_1 x^2(t)x^3(t)+ax^2(t)+c_{31}x^1 \\ D_t^q x^2(t)=\alpha_2 x^1(t)x^3(t)+bx^1(t)+c_{31}x^2, \\ D_t^q x^3(t)=\alpha_3 x^1(t)x^2(t)+c_{32}\left(x^3-m\right) \end{cases} \qquad q\in(0,1) \tag{3.6}$$

where $c_{31},c_{32}\in\mathbf{R}$ are real constants.

The Jacobian matrix of the perturbed fractional Euler top system (3.6) for e_3^m is

$$J_{a,b}(x,c_{31},c_{32})=\begin{pmatrix} c_{31} & \alpha_1 x^3+a & \alpha_1 x^2 \\ \alpha_2 x^2+b & c_{31} & \alpha_2 x^1 \\ \alpha_3 x^2 & \alpha_3 x^1 & c_{32} \end{pmatrix}.$$

Proposition 3.6. *Let be the perturbed fractional Euler top system* (3.6).

(i) Let $(\alpha_1 m+a)(\alpha_2 m+b)<0$ *and* $m\in\mathbf{R}$.

(1) If $c_{31}<0, c_{32}<0,$ *then* e_3^m *is asymptotically stable* $(\forall)q\in(0,1)$;

(2) If $c_{31} > 0, c_{32} < 0,$ *then* e_3^m *is asymptotically stable* $(\forall)q \in (0, q_3),$ *where*
$q_3 = \frac{2}{\pi} | \arctan \frac{\sqrt{| (\alpha_1 m + a)(\alpha_2 m + b) |}}{| c_{31} |} |$ *and it is stable for* $q = q_3.$

(ii) Let $(\alpha_1 m + a)(\alpha_2 m + b) > 0.$ *If* $c_{31} < 0, c_{32} < 0$ *and* $(\alpha_1 m + a)(\alpha_2 m + b) < c_{31}^2,$ *then* e_3^m *is asymptotically stable for all* $q \in (0,1).$

Proof. Let be $p_3(\lambda)$ the characteristic polynomial of matrix $J_{a,b}(e_3^m, c_{31}, c_{32}),$ where $p_3(\lambda) = -(\lambda - c_{32})[(\lambda - c_{31})^2 - (\alpha_1 m + a)(\alpha_2 m + b)].$ The roots of equation $p_3(\lambda) = 0$ are $\lambda_1 = c_{32},$ $\lambda_{2,3} = c_{31} \pm \sqrt{| d |},$ where $d = (\alpha_1 m + a)(\alpha_2 m + b).$

(i) **Case** $d < 0.$ Then $\lambda_1 = c_{32},$ $\lambda_{2,3} = c_{31} \pm i\sqrt{| d |}.$

(i.1) We suppose that $c_{32} < 0$ and $c_{31} < 0.$ Then $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0.$ Since $| \arg(\lambda_i) | = \pi > \frac{q\pi}{2},\ i = \overline{1,3}$ for all $q \in (0,1),$ by Proposition 3.2(i), it implies that e_3^m is locally asymptotically stable for all $m \in \mathbf{R}.$

(i.2) For $c_{32} < 0$ and $c_{31} > 0$ $c_{11} < 0$ and $c_{12} > 0,$ we have $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) > 0.$ Applying Proposition 3.2(i), e_3^m is locally asymptotically stable, for $0 < q < q_3,$ where $q_3 = \frac{2}{\pi} | \arctan \frac{\sqrt{| d |}}{| c_{31} |} |.$ Therefore, the assertion (i.2) holds. If $q = q_3,$ then e_3^m is stable. For $q_3 < q < 1,$ e_3^m is unstable for all $m \in \mathbf{R}.$

(i.3) Let $c_{32} > 0$ and $c_{31} \in \mathbf{R}.$ then $\lambda_1 > 0$ and $J_{a,b}(e_3^m, c_{31}, c_{32})$ has at least a positive eigenvalue. Hence, e_3^m is unstable for all $(\forall)m \in \mathbf{R}.$ Therefore, the assertions (i) hold.

(ii) **Case** $d > 0.$ Then $\lambda_1 = c_{32},$ $\lambda_{2,3} = c_{31} \pm \sqrt{| d |}.$ In this case, $\lambda_i, i = \overline{1,3}$ are all negative if and only if $c_{31} < 0, c_{32} < 0$ and $(\alpha_1 m + a)(\alpha_2 m + b) < c_{31}^2.$ It follows that e_3^m is asymptotically stable for all $q \in (0,1).$ □

If in (3.6) we take $\alpha = (a_3 - a_2, a_1 - a_3, a_2 - a_1),$ then one obtains the *perturbed fractional rigid body equations with two linear controls for* $e_3^m = (0,0,m).$

Corollary 3.3. *Let be the perturbed fractional rigid body equations with two linear controls for* $e_3^m = (0,0,m).$ *If* $c_{31} < 0, c_{32} < 0,$ *then* e_3^m *is asymptotically stable for all* $m \in \mathbf{R}^*$ *satisfying the condition* $\big((a_3 - a_2)m + a\big)\big((a_1 - a_3)m + b\big) < 0$ *and* $q \in (0,1).$

Proof. Since $(a_3 - a_2)(a_1 - a_3) < 0,$ the assertion follows from Proposition 3.6 (i). □

Example 3.2. *(i)* Let be the perturbed fractional Rikitake-Hamilton system for $e_1^m = (m, 0, \beta).$ For $c_{11} = -1, c_{12} = -2, \beta = 1, m = 0.75,$ the conditions of Corollary 3.1 are achieved. Then $e_1 = (-0.75, 0, 1)$ is asymptotically stable $(\forall)q \in (0,1).$

(ii) Let be perturbed fractional Rabinovich system for $e_2^m = (0, m, -\beta).$ If we select $c_{21} = -2, c_{22} = -3, \beta = 0.5$ and $m = 0.75,$ then the conditions of Corollary 3.2 are achieved. Then $e_2 = (0, 0.75, 0.5)$ is asymptotically stable for $q \in (0,1).$ □

IV. NUMERICAL INTEGRATION OF FRACTIONAL MODEL (4.1)

The fractional differential systems $(3.4)-(3.6)$ can be written in the general form:

$$\begin{cases} D_t^q x^1 = \alpha_1 x^2 x^3 + a x^2 + k_1\left(x^1 - x_e^1\right) \\ D_t^q x^2 = \alpha_2 x^1 x^3 + b x^1 + k_2\left(x^2 - x_e^2\right), \\ D_t^q x^3 = \alpha_3 x^1 x^2 + k_3\left(x^3 - x_e^3\right) \end{cases} \qquad q \in (0,1) \qquad (4.1)$$

where $k_1, k_2, k_3 \in \mathbf{R}$ are real constants.

The fractional dynamics (4.1) is called the *perturbed fractional Euler top system with two linear controls for the equilibrium state* $x_e = (x_e^1, x_e^2, x_e^3)$.

For example, if in the system (4.1) we take $k_1 = c_{11}, k_{2,3} = c_{12}$ and $x_e = e_1^m$ one obtains the perturbed fractional Euler top system (3.4) for the equilibrium state e_1^m.

Consider the fractional differential equations

$$\begin{cases} D_t^q x^i(t) = F_i(x^1(t), x^2(t), x^3(t)), & t \in (0,\tau), \quad i = \overline{1,3}, \quad q \in (0,1) \\ \quad x(0) = \left(x^1(0), x^2(0), x^3(0)\right), & \end{cases} \qquad (4.2)$$

where

$$\begin{cases} F_1(x(t)) = \alpha_1 x^2(t) x^3(t) + a x^2(t) + k_1\left(x^1(t) - x_e^1\right) \\ F_2(x(t)) = \alpha_2 x^1(t) x^3(t) + b x^1(t) + k_2\left(x^2(t) - x_e^2\right). \\ F_3(x(t)) = \alpha_3 x^1(t) x^2(t) + k_3\left(x^3(t) - x_e^3\right) \end{cases}$$

Since the functions $F_i(t), i = \overline{1,3}$ are continuous, then the initial value problem (4.2) is equivalent to nonlinear Volterra integral equation of the second order [5], which is given as follows:

$$\dot{x}^i(t) = x_0^i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F_i(x^1(s), x^2(s), x^3(s)) ds, \qquad i = \overline{1,3}, \quad q > 0. \qquad (4.3)$$

Diethelm et al. used the predictor-corrector scheme [4], based on the *Adams-Moulton algorithm* to integrate the equations (4.3).

We apply this scheme to fractional system (4.1). For this, let $h = \dfrac{\tau}{N}, t_n = nh$ for $n = 0,1,\ldots,N.$

The perturbed fractional system (4.1) can be integrated as follows:

$$\begin{cases} x^i[n+1]) = x_0^i + \dfrac{h^q}{\Gamma(q+2)}\left(\displaystyle\sum_{j=0}^{n} a[j,n+1] F_i(x^1[j], x^2[j], x^3[j])\right) + \\ \qquad + F_i(x_p^1[n+1], x_p^2[n+1], x_p^3[n+1]), & i = \overline{1,3}, \qquad (4.4) \\ x_p^i[n+1]) = \dfrac{h^q}{q\Gamma(q)}\left(\displaystyle\sum_{j=0}^{n} b[j,n+1] F_i(x^1[j], x^2[j], x^3[j])\right) \end{cases}$$

where:

$$\begin{cases} a[0,n+1] = n^{q+1}\text{-}(n\text{-}q)(n+1)^q \\ a[j,n+1] = (n-j+2)^{q+1} + (n\text{-}j)^{q+1} - 2(n-j+1)^{q+1}, & j = \overline{1,n}, \\ b[j,n+1] = (n-j+1)^q - (n\text{-}j)^q, & j = \overline{0,n}. \end{cases}$$

The above scheme given by the relations (4.4) is called the *Moulton-Adams algorithm for perturbed fractional system* (4.1) [4].

The error estimate for the algorithm described by (4.4) is

$$\max_{0\leq j\leq N}\left\{\left(x^i[j]-x_p^i[j]\right)\mid i=\overline{1,3}\right\}=O(h^{q+1}).$$

Let us we apply the algorithm (4.4) and software Maple 11, to integrate two perturbed fractional Euler top systems with two linear controls of type (4.1). For this, we take $h=0.01, \varepsilon=0.01, N=500, t=502$ and the initial conditions $x(0)=(x^1(0),x^2(0),x^3(0)),$ where $x^i(0)=\varepsilon+x_e^i, i=\overline{1,3}.$

These considerations are exemplified in the following cases.

(1) Let be the fractional model (4.1) for the equilibrium state $e_1^m=(m,0,\beta)$ associated to Rikitake-Hamilton system (2.4) with $m=-0.75$ and $\beta=1,$ which has discussed in Example 3.2. *(i)*. In the relations (4.4) we take:

$\alpha_1=-0.5,\alpha_2=1,\alpha_3=-1,a=1,b=-1,k_1=-1,k_2=k_3=-2,x_e^1=-0.75,x_e^2=0$ and $x_e^3=1.$

In the coordinate system $Ox^1x^2x^3,$ the orbits of solutions of fractional Rikitake-Hamilton model (4.1) for the equilibrium state $e_1=(-0.75,0,1)$ have the representations given in the figures Fig.1 (for $q=0.75$) and Fig. 2 (for $q=1$).

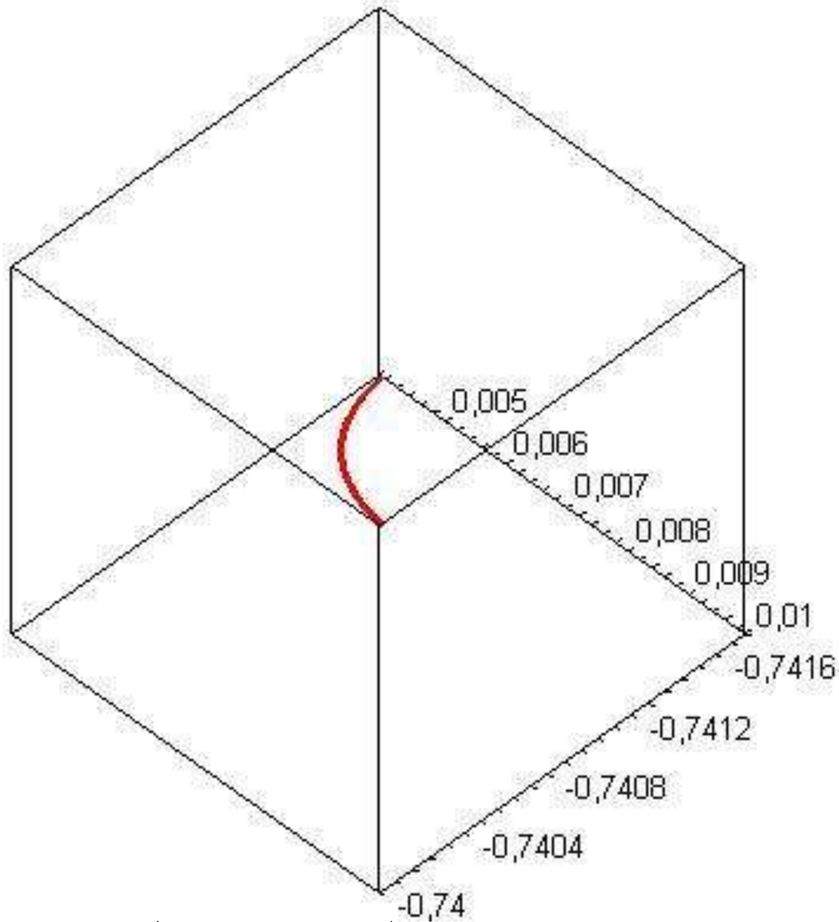

Fig.1. *Orbits* $\left(x^1(n),x^2(n),x^3(n)\right)$ *of Rikitake model for* $q=0.75$

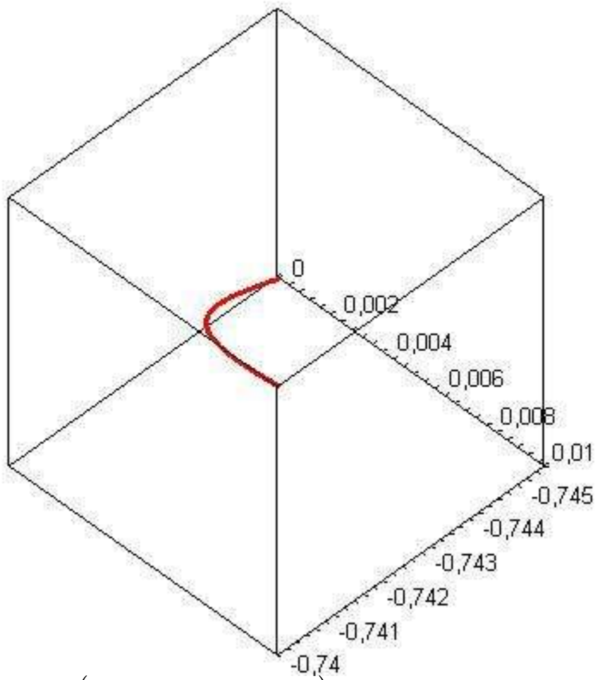

Fig. 2. *Orbits* $\left(x^1(n), x^2(n), x^3(n)\right)$ *of Rikitake model for* $q = 1$

(2) Let be the fractional model (4.1) for the equilibrium state $e_2^m = (0, m, -\beta)$ associated to Rabinovich system (2.5) with $m = 0.75$ and $\beta = -0.5$, which has presented in Example 3.2. *(i).* In the relations (4.4) we take:

$\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1, a = -0{,}5, b = -0.5, k_1 = k_3 = -2, k_2 = -3, x_e^1 = 0, x_e^2 = 0.75$ and $x_e^3 = 0.5$.

In the coordinate system $Ox^1x^2x^3$, the orbits of solutions of fractional Rabinovich model (4.1) for the equilibrium state $e_2 = (0, 0.75, 0.5)$ have the representations given in the figures Fig.3 (for $q = 0.75$) and Fig. 4 (for $q = 1$).

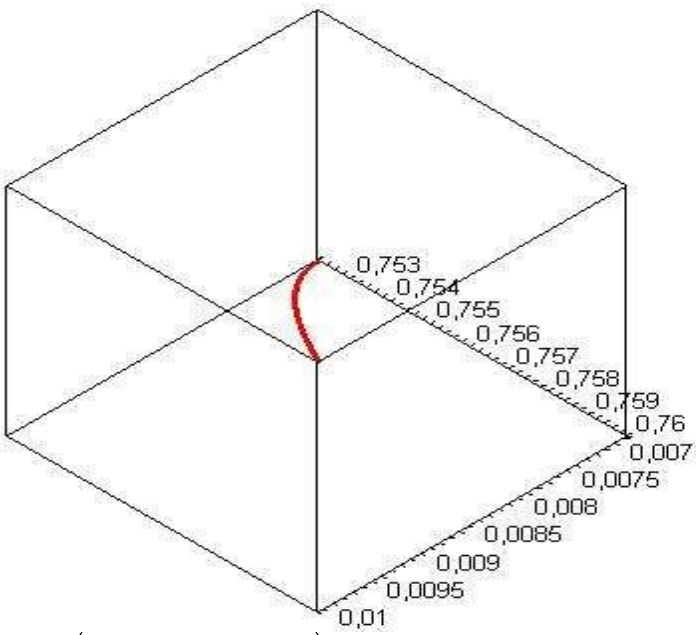

Fig.3. *Orbits* $\left(x^1(n), x^2(n), x^3(n)\right)$ *of Rabinovich model for* $q = 0.75$

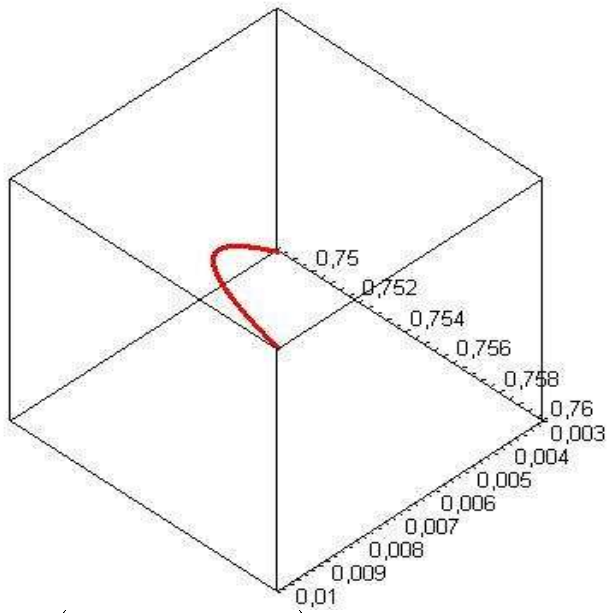

Fig. 4. *Orbits* $\left(x^1(n), x^2(n), x^3(n)\right)$ *of Rabinovich model for* $q=1$

The numerical simulations show the validity of the theoretical analysis.

V. CONCLUSIONS.

The dynamics of the fractional Euler top system with two control parameters (3.1) was studied. The analysis of the fractional stability of for the perturbed fractional model associated to system (3.1) has investigated. Finally, the numerical simulations for solutions of perturbed fractional Rikitake-Hamilton system and perturbed fractional Rabinovich system are given. □

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