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A posteriori error estimation for incompressible flow problem

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ABSTRACT

This paper describes numerical solutions of incompressible Navier-Stokes equations. It includes algorithms for discretization by finite element methods and a posteriori error estimation of the computed solutions.

A numerical experiment on the driven cavity flow is given to demonstrate the effectiveness of the error estimate. We compare the result with the solution from ADINA system as well as with values from other simulations.

Keywords **- Navier-Stokes Equations, Finite Element Method, A posteriori error estimation, Adina system.**

I. INTRODUCTION

This paper describes a numerical solution of partial differential equations (PDEs) that are used to model steady incompressible fluid flow. For the equations, we offer a choice of two-dimensional domains on which the problem can be posed, along with boundary conditions and other aspects of the problem, and a choice of finite element discretizations on a quadrilateral element mesh.

The plan of the paper is as follows. The model problem is described in section II, followed by a posteriori error bounds of the computed solution in section III and numerical experiments are carried out in section VI.

II. INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

We consider the steady-state Navier-Stokes equations for the flow of a Newtonian incompressible viscous fluid with constant viscosity:

$$\begin{cases} -\nu\nabla^2\vec{u}+\vec{u}.\nabla\vec{u}+\nabla p=\vec{f}, & \text{sur } \Omega \\ \nabla.\vec{u}=0, & \text{sur } \Omega \\ \vec{u}=\vec{0}, & \text{sur } \Gamma \end{cases} \qquad (1)$$

Where $\nu > 0$ is a given constant called the kinematic viscosity.
$\vec{u}$ is the fluid velocity, p is the pressure field, ∇ is the gradient and $\nabla\cdot$ is the divergence operator.

The boundary value problem that is considered is the system (1) posed on two or three-dimensional domain Ω.

We define the spaces:

$$V = H_0^1(\Omega)\times H_0^1(\Omega), \qquad (2)$$

and

$$W=\left\{q\in L^2(\Omega): \int_\Omega q(x)dx=0\right\}, \qquad (3)$$

Let the bilinear forms $a: V\times V \to IR$, $b: V\times W \to IR$, $d: W\times W \to IR$, and the trilinear form $c: V\times V\times V \to IR$

$$a(u,v)=\nu\int_\Omega \nabla u.\nabla v dx,\quad b(v,q)=-\int_\Omega (q\nabla\cdot\upsilon)dx,\quad d(p,q)=\int_\Omega pq dx, \qquad (4)$$

$$c(z,u,\upsilon)=\int_\Omega (z\cdot\nabla u)\cdot\upsilon\,. \qquad (5)$$

These inner products induce norms on V and W denoted by $\|\upsilon\|_V$ and $\|q\|_W$ respectively.

$$\|v\|_V = a(v,v)^{\frac{1}{2}} \quad \forall v\in V, \qquad (6)$$

$$\|q\|_W = d(q,q)^{\frac{1}{2}} \quad \forall q\in W. \qquad (7)$$

Given the continuous functional $l: V\to IR$

$$l(\upsilon)=\int_\Omega f\cdot\upsilon\, dx, \qquad (8)$$

Then the standard weak formulation of the Navier-Stokes flow problem (1) is the following:

Find $(u,p)\in V\times W$ such that

$$a(u,\upsilon)+b(\upsilon,p)+c(u,u,\upsilon)+b(u,q)=l(q), \text{ for all } (\upsilon,q)\in V\times W. \qquad (9)$$

Let the subspace of divergence-free velocities be given by

$$V_{E_0}=\{z\in V; z\cdot n=0 \text{ sur } \partial\Omega \text{ et } \nabla\cdot z=0 \text{ sur } \Omega\}. \qquad (10)$$

The convection term is skew-symmetric: $c(z,u,\upsilon)=-c(z,\upsilon,u)$; over V_{E_0},this mean that

$$c(z,u,u)=0 \quad \forall z\in V_{E_0} \qquad (11)$$

The problem (9) is known [4] to possess a unique solution whenever the data is sufficiently small. In particular, if

$$l(\upsilon) \leq \theta \frac{\nu^2}{C} |\upsilon|_{H^1(\Omega)} \quad \forall \upsilon \in \mathrm{V}, \tag{12}$$

For some fixed $\theta \in [0,1)$ then there is a unique solution $u \in V$ satisfying

$$|u|_{H^1(\Omega)} \leq \theta \frac{\nu}{C}. \tag{13}$$

Let P be a regular partitioning of the domain Ω into the union of N subdomains K such that

- $N < \infty$,
- $\overline{\Omega} = \bigcup_{K \in P} \overline{K}$,
- $K \cap J$ is empty whenever $K \neq J$,
- Each K is a convex Lipschitzian domain with piecewise smooth boundary ∂K .

The common boundary between subdomains K and J is denoted by

$$\Gamma_{KJ} = \partial K \cap \partial J . \tag{14}$$

The finite element subspaces X and M are constructed in the usual manner so that the inclusion $X \times M \subset V \times W$ holds.

The finite element approximation to (9) is then

Find (u^X, p^M) such that

$$a(u^X, \upsilon^X) + b(\upsilon^X, p^M) + c(u^X, u^X, \upsilon^X) + b(u^X, q^M) = l(q^M) \tag{15}$$

For all $(\upsilon^X, q^M) \in V \times W$.

Let $(e, E) \in V \times W$ be the error in the finite element approximation, $e = u - u^X$ and $E = p - p^M$ and define $(\phi, \psi) \in V \times W$ to be the Ritz projection of the modified residuals

$$a(\phi, \upsilon) + d(\psi, q) = a(e, \upsilon) + b(\upsilon, E) + b(e, q) + D(u, u^X, \upsilon), \tag{16}$$

for all $(\upsilon, q) \in V \times W$,

where

$$D(u, u^X, \upsilon) = c(u, u, \upsilon) - c(u^X, u^X, \upsilon).$$

Theorem 2.1. Let (13) hold. Then there exist positive constants K_1 and K_2 such that

$$K_1\left(\|\phi\|_V^2 + \|\psi\|_W^2\right) \leq \left\|u - u^X\right\|_V^2 + \left\|p - p^M\right\|_W^2 \leq K_2\left(\|\phi\|_V^2 + \|\psi\|_W^2\right) \tag{17}$$

Proof. See T.J. Oden, W. Wu, and M. Ainsworth [16].

III. A POSTERIORI ERROR ANALYSIS

The local velocity space on each subdomain $K \in P$ is

$$V_K = \left\{\upsilon \in H^1(K) \times H^1(K) : \upsilon = 0 \,\mathrm{sur}\, \partial\Omega \cap \partial K\right\} \tag{18}$$

And the pressure space is

$$W_K = L^2(K) \tag{19}$$

Let the bilinear forms

$a_K : V_K \times V_K \to IR$, $b_K : V_K \times W_K \to IR$, $d_K : W_K \times W_K \to IR$,

and the trilinear form $c_K : V_K \times V_K \times V_K \to IR$

$$a_K(u, \upsilon) = \nu \int_K \nabla u \cdot \nabla \upsilon, \; b_K(\upsilon, q) = -\int_K q(\nabla \cdot \upsilon),$$

$$d_K(p, q) = -\int_K pq \tag{20}$$

$$c_K(z; u, \upsilon) = \int_K (z \cdot \nabla u) \cdot \upsilon \tag{21}$$

Given the continuous functional $l_K : V_K \to IR$

$$l_K(\upsilon) = \int_K f \cdot \upsilon \, \mathrm{d}x \tag{22}$$

Hence for $\upsilon, w \in V$ and $q \in W$ we have

$$a(\upsilon, w) = \sum_{K \in P} a_K(\upsilon_K, w_K) \tag{23}$$

$$b(\upsilon, q) = \sum_{K \in P} b_K(\upsilon_K, q_K) \tag{24}$$

$$c(z, u, \upsilon) = \sum_{K \in P} c_K(z_K, u_K, \upsilon_K) \tag{25}$$

$$l(\upsilon) = \sum_{K \in P} l_K(\upsilon_K) \tag{26}$$

The broken velocity space $V(P)$ is defined by

$$V(P) = \prod_{K \in P} V_K \tag{27}$$

And the broken pressure space is defined by

$$W(P) = \left\{q \in \prod_{K \in P} W_K : \int_\Omega q(x)\,\mathrm{d}x = 0\right\} \tag{28}$$

Examining the previous notations reveals that

$$W(P) = W \tag{29}$$

We consider the space of continuous linear functional τ on $V(P)\times W(P)$ that vanish on the space $V\times W$ Therefore, let $H(div,\Omega)$ denote the space

$$H(div,\Omega)=\left\{A\in L^2(\Omega)^{2\times 2} : divA\in L^2(\Omega)^2\right\} \quad (30)$$

Equipped with norm

$$\|A\|_{H(div,\Omega)}=\left\{\|A\|^2_{L^2(\Omega)}+\|divA\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} \quad (31)$$

Theorem 3.1. A continuous linear functional τ on the space $V(P)\times W(P)$ vanishes on the space $V\times W$ if and only if there exists $A\in H(div,\Omega)$ such that

$$\tau[(\upsilon,q)]=\sum_{K\in P}\oint_{\partial K} n_K\cdot A\upsilon_K ds, \quad (32)$$

Where n_K denotes the unit outward normal vector on the boundary of K.

Proof. See M. Ainsworth and J. Oden [5].

It will be useful to introduce the stress like tensor $\sigma(\upsilon,q)$ formally defined to be

$$\sigma_{ij}=\nu\frac{\partial\upsilon_i}{\partial x_j}-q\delta_{ij} \quad (33)$$

where δ_{ij} is the Kronecker symbol.

In order to define the value of the normal component of the stress on the interelement boundaries it is convenient to introduce notations for the jump on Γ_{KJ} :

$$[\![n\cdot\sigma(\upsilon^X,q^M)]\!]=n_K\cdot\sigma(\upsilon^X_K,q^M_K)+n_J\cdot\sigma(\upsilon^X_J,q^M_J) \quad (34)$$

Furthermore, an averaged normal stress on is defined as

$$\langle n_K\cdot\sigma(\upsilon^X,q^M)\rangle=\begin{bmatrix}\alpha^{(1)}_{KJ} & 0\\ 0 & \alpha^{(2)}_{KJ}\end{bmatrix}n_K\cdot\sigma(\upsilon^X_K,q^M_K)+\begin{bmatrix}\alpha^{(1)}_{JK} & 0\\ 0 & \alpha^{(2)}_{JK}\end{bmatrix}n_K\cdot\sigma(\upsilon^X_J,q^M_J) \quad (35)$$

where $\alpha^{(i)}_{KJ}:\Gamma_{KJ}\rightarrow IR$ are smooth (polynomial) functions. Naturally, should the stress be continuous then it is required that the averaged stress coincide with this value. Therefore, on Γ_{KJ},

$$\begin{bmatrix}\alpha^{(1)}_{KJ} & 0\\ 0 & \alpha^{(2)}_{KJ}\end{bmatrix}+\begin{bmatrix}\alpha^{(1)}_{JK} & 0\\ 0 & \alpha^{(2)}_{JK}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad (36)$$

The notation [[.]] is also used to define jumps in the elements of $V(P)$ between subdomains.

Define

$$[\![\upsilon]\!]=\begin{cases}\upsilon_K-\upsilon_J \ , \ K>J \ ,\\ \upsilon_J-\upsilon_K \ , \ K<J \ ,\end{cases} \quad (37)$$

and

$$[\![n]\!]=\begin{cases}n_K-n_J \ , \ K>J \ ,\\ n_J-n_K \ , \ K<J \ .\end{cases} \quad (38)$$

The following identity, valid for $\upsilon\in V(P)$ is readily verified:

$$\sum_{K\in P}\oint_{\partial K}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot\upsilon ds=\sum_{\Gamma_{KJ}}\int_{\Gamma_{KJ}}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot[\![\upsilon]\!]ds \quad (39)$$

Lemma 3.1. Under the above notations and conventions, there exists $\bar{\mu}\in H(div,\Omega)$ such that for all $(w,q)\in V(P)\times W(P)$

$$\bar{\mu}[(w,q)]=\sum_{\Gamma_{KJ}}\int_{\Gamma_{KJ}}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot[\![w]\!]ds \quad (40)$$

Proof. See M. Ainsworth and J. Oden [14].

Summarizing, we have shown the following :

Theorem 3.2. Let $J_K:V_K\rightarrow IR$ be a quadratic functional

$$\begin{aligned}J_K(w_K)=&\frac{1}{2}a_K(w_K,w_K)-l_K(w)+a_K(u^X,w_K)+b_K(w_K,p^M)\\&+C_K(u^X,u^X,w_K)-\oint_{\partial K}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot w_K ds.\end{aligned} \quad (41)$$

Then

$$\|\phi\|^2_V+\|\psi\|^2_W\leq\sum_{K\in P}\left(-2\inf_{w_K\in V_K}J_K(w_K)+d_K(\nabla\cdot u^X_K,\nabla\cdot u^X_K)\right). \quad (42)$$

The analysis has leads to problems on each subdomain of the form

$$\inf_{w_K\in V_K}J_K(w_K).$$

Suppose for a moment that the minimum exists, then the minimizing element is characterized by finding $\phi_k\in V_K$ such that

$$a(\phi_k,\upsilon)=l_K(\upsilon)-a_K(u^X,\upsilon)-b_K(\upsilon,p^M)-C_K(u^X,u^X,\upsilon)+\oint_{\partial K}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot\upsilon ds \quad (43)$$

for all $\upsilon\in V_K$.

The necessary and sufficient conditions for the existence of a minimum are that the data satisfy the following compatibility for equilibration condition:

$$0=l_K(\theta)-a_K(u^X,\theta)-b_K(\theta,p^M)-C_K(u^X,u^X,\theta)+\oint_{\partial K}\langle n_K\cdot\sigma(u^X,p^M)\rangle\cdot\theta ds \quad (44)$$

For all $\theta \in Ker[a, V_K]$,

where

$$Ker[a, V_K] = \{\theta \in V_K : a_K(w, \theta) = 0 \ \forall w \in V_K\}. \quad (45)$$

When the subdomain K lies on the boundary $\partial\Omega$ the local problem (43) will be subject to a homogeneous Dirichlet condition on a portion of their boundaries and thus will be automatically well posed. However, elements away from the boundary are subject to pure Neumann conditions and the null space of the operator $a(\cdot,\cdot)$ will contain the rigid motions

$$Ker[a, V_K] = Span\{\theta_1, \theta_2\}, \quad (46)$$

and

$$\theta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \theta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (47)$$

We shall be able to construct data which satisfy the equilibration condition (44). First we define

$$\begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} = \begin{bmatrix} \alpha_{KJ}^{(1)} & 0 \\ 0 & \alpha_{KJ}^{(2)} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (48)$$

So that consistency condition (36) becomes

$$\begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} + \begin{bmatrix} \lambda_{JK}^{(1)} & 0 \\ 0 & \lambda_{JK}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (49)$$

The averaged interelement stress may be rewritten

$$\langle n_K \cdot \sigma(\upsilon^X, q^M) \rangle = \langle n_K \cdot \sigma(\upsilon^X, q^M) \rangle_{\frac{1}{2}} + [[n \cdot \sigma(\upsilon^X, q^M)]] \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} \quad (50)$$

Where $\langle n_K \cdot \sigma(\upsilon^X, q^M) \rangle_{\frac{1}{2}}$ denotes the interelement averaged stress obtained using the symmetrical weighting corresponding to $\alpha = \frac{1}{2}$. The equilibration condition then becomes

$$\begin{aligned} &l_K(\theta) - a_K(u^X, \theta) - b_K(\theta, p^M) - C_K(u^X, u^X, \theta) + \int_{\partial K} \langle n_K \cdot \sigma(u^X, p^M) \rangle \cdot \theta ds \\ &= -\sum_{J \in P} \int_{\Gamma_{KJ}} [[n \cdot \sigma(u^X, p^M)]] \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} \cdot \theta ds, \end{aligned} \quad (51)$$

For all $\theta \in Ker[a, V_K]$.

Let $\{V_A\}$ be chosen so that $Span\{V_A\} \times Span\{V_A\} \subset V$ and scaled so that

$$\sum_A V_A(x) \equiv 1. \quad (52)$$

The relation (52) must hold at all points x contained in elements which do not interest the boundary of the domain.

The functions $\lambda_{KJ}^{(k)} : \Gamma_{KJ} \to IR$ are chosen to be of the form

$$\lambda_{KJ}^{(k)}(s) = \sum_A \lambda_{KJ,A}^{(k)} V_A(s), \quad (53)$$

Where $\lambda_{KJ,A}^{(k)}$ are constants to be determined. Owing the constraint (49), it is required that

$$\lambda_{KJ,A}^{(k)} + \lambda_{JK,A}^{(k)} = 0, \quad (54)$$

for each A.

Lemma 3.2. Suppose that for each V_A the constants $\lambda_{KJ,A}^{(k)}$ can be chosen to satisfy

$$-\sum_{J \in P} \lambda_{KJ,A}^{(k)} \rho_{KJ,A}^{(k)} = b_{K,A}^{(k)} \quad (55)$$

for k=1, 2, where

$$\begin{aligned} b_{K,A}^{(k)} &= l_K(V_A\theta_k) - a_K(u^X, V_A\theta_k) - b_K(V_A\theta_k, p^M) - C_K(u^X, u^X, V_A\theta_k) \\ &+ \int_{\partial K} V_A(s) \langle n_K \cdot \sigma(u^X, p^M) \rangle \cdot \theta_k ds, \end{aligned} \quad (56)$$

and

$$\rho_{KJ,A}^{(k)} = \int_{\Gamma_{KJ}} [[n \cdot \sigma(\upsilon^X, q^M)]] \cdot \theta_k ds \quad (57)$$

Then

$$0 = l_K(\theta) - a_K(u^X, \theta) - b_K(\theta, p^M) - C_K(u^X, u^X, V_A\theta_k) + \int_{\partial K} \langle n_K \cdot \sigma(u^X, p^M) \rangle \cdot \theta ds, \quad (58)$$

for all $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$.

Proof. The result follows immediately by forming appropriate linear combination of (55), and using (53) and (49). Summarizing and incorporating the results of section 3 we have

Theorem 3.3. Let the conditions of Theorem 2.1 hold. Then there exists a constant C such that

$$\|(e, E)\|_*^2 \le C \sum_{K \in P} \eta_K^2 \quad (59)$$

Where

$$\eta_K = \{a_K(\phi_K, \phi_K) + d_K(\nabla \cdot u^X, \nabla \cdot u^X)\}^{\frac{1}{2}}, \quad (60)$$

We define the global error estimator

$$\eta = \left(\sum_{K \in P} \eta_K^2 \right)^{\frac{1}{2}} \quad (61)$$

IV. NUMERICAL SIMULATIONS

In this section some numerical results of calculations with finite element Method and ADINA system will be presented. Using our solver, we run the test problem driven cavity flow [1, 6, 7, 8, 9, 10].

$\{y = 1; -1 \le x \le 1 \mid u_x = 1 - x^4\}$ a leaky cavity.

The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

The solution shown in figure 1 corresponds to a Reynolds number of 100. The particles in the body of the fluid move in a circular trajectory.

The profiles of the u-velocity component along the vertical centerline and the v-velocity component along the horizontal centerline are shown in Figure 2 for Re=1000. In this figure, we have also included numerical predictions from [6] and ADINA system. There is excellent agreement between the computed results, those published in [6] and the results computed with ADINA system.

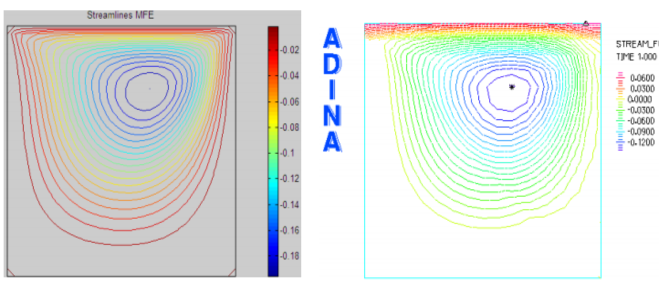

Fig.1. Uniform streamline plot with FE (left), and uniform streamline plot computed with ADINA system (right) using $Q_1 - P_0$ approximation, a 32×32 square grid and Reynolds number Re=100.

This is a classic test problem used in fluid dynamics, known as driven-cavity flow. It is a model of the flow in a square cavity with the lid moving from left to right. Let the computational model:

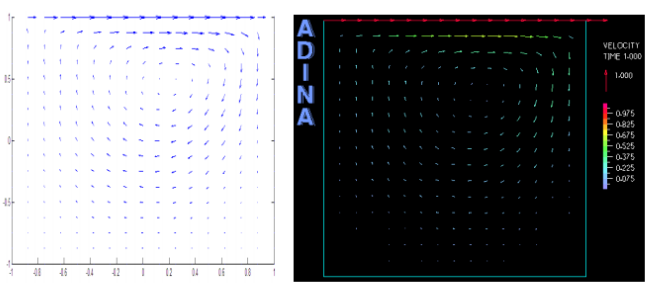

Fig.2. Velocity vectors solution by FE (left) and velocity vectors solution (right) computed with ADINA system with a 32×32 square grid and Re= 100.

Table 1. Estimated errors for leaky driven cavity for the flow with Reynolds number Re =100.

	$\|u - u^X\|_V$	η
8×8	8.704739×10^{-2}	1.720480×10^{0}
16×16	3.115002×10^{-2}	1.084737×10^{0}
32×32	9.545524×10^{-3}	5.919904×10^{-1}
64×64	2.676623×10^{-3}	3.160964×10^{-1}

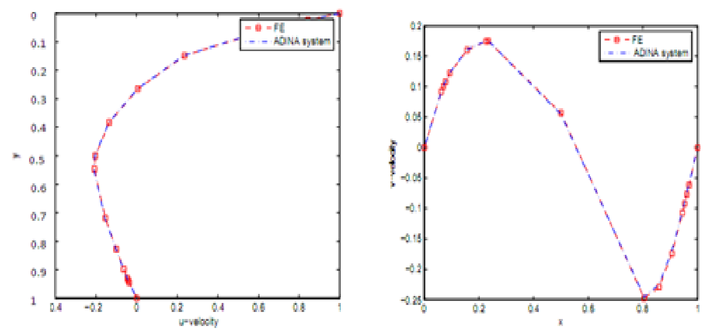

Fig.3. Velocity component u at vertical centerline (left plot), and the velocity component v at horizontal centerline (right plot) with a 129×129 square grid and Reynolds number Re=1000.

Figure 3 shows the velocity profiles for lines passing through the geometric center of the cavity.

These features clearly demonstrate the high accuracy achieved by the proposed finite element method for solving the Navier-Stokes equations in the lid-driven squared cavity.

V. CONCLUSION

We were interested in this work in the numeric solution for two dimensional partial differential equations modeling (or arising from) steady incompressible fluid flow. It includes algorithms for discretization by finite element methods and a posteriori error estimation of the computed solutions.

Our results agree with Adina system.

Numerical experiments were carried out and compared with satisfaction with other numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.

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