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B-Continuity in Peterson graph and power of a Cycle

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ABSTRACT: *A graph G is k-colorable if G has a proper vertex coloring with k colors. The chromatic number $\chi(G)$ is the minimum number k such that G is k-colorable. A b-coloring of a graph with k colors is a proper coloring in which each color class contains a color dominating vertex. The largest positive integer k for which G has a b-coloring with k colors is the b-chromatic number of G, denoted by b(G). The b-spectrum $S_b(G)$ of G, is defined as the set of all integers k at which G is b-colorable with k colors. A graph G is b-continuous if its b-spectrum equals $[\chi(G), b(G)]$. In this paper, we prove that the Peterson graph and the power of a cycle are b-continuous. Also, we prove that the Cartesian product of two cycles $C_m \square C_n$ is b-continuous when m and n are multiples of 5. In this case, we give the color classes of b-coloring with k colors for each k with $\chi(G) \leq k \leq b(G)$.*

Keywords: *b-coloring, b-continuous. Peterson graph, power of a cycle.*

I. INTRODUCTION

A k-vertex coloring of a graph G is an assignment of k colors 1, 2… k, to the vertices. The coloring is proper if no two distinct adjacent vertices share the same color. A graph G is k-colorable if it has a proper k-vertex coloring [5]. The chromatic number $\chi(G)$ is the minimum number k such that G is k-colorable. Color of a vertex v is denoted by c(v). A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. In other words, each color class contains a color dominating vertex (a vertex which has a neighbor in all the other color classes). The b-chromatic number b(G) is the largest integer k such that G admits a b-coloring with k colors.

The b-spectrum $S_b(G)$ of G is defined by $S_b(G) = \{ k \in N : \chi(G) \leq k \leq b(G)$ and G is b-colorable with k colors $\}$. A graph G is b-continuous if $S_b(G) = [\chi(G), b(G)]$.

El-Sahili[3] conjectured that every d-regular graph with girth at least 5 has a b-coloring with d + 1 colors.

Marko Jaovac and Sandi Klavzar[7] disproved this conjecture and they proved the following:
`Peterson graph is a 3-regular graph with girth 5 and b-chromatic number 3'.

In section 3, we prove this result in another way. Saeed Shaebani[9] proved that some of the Kneser graphs are b-continuous. Further, they gave some special conditions for graphs to be b-continuous.

Here we list out some of the necessary definitions. For a graph G, and for any vertex v of G, the neighborhood of v is the set $N(v) = \{ u \in V(G) : (u, v) \in E(G) \}$ and the degree of v is $deg(v) = deg_G(v) = |N(v)|$[4]. $\Delta(G)$ denotes the maximum degree of a vertex in G. Note that every graph G satisfies $b(G) \leq \Delta(G)+1$.

A graph is a power of cycle, denoted C_n^k, if $V(C_n^k) = \{ v_0(= v_n), v_1, v_2, \ldots, v_{n-1} \}$ and $E(C_n^k) = E^1 \cup E^2 \cup \ldots \cup E^k$, where $E^i = \{ (v_j, v_{(j+i)(\text{mod } n)}) : 0 \leq j \leq n-1\}$ and $k \leq (n-1)/2$[1]. Note that C_n^k is a 2k-regular graph and $k \geq 1$. We take $(v_0, v_1, v_2, \ldots, v_{n-1})$ to be a cyclic order on the vertex set of G, and perform modular operations on the vertex indexes [1].

The Cartesian product $G \square H$ of two graphs G and H is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{ ((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(G)$ with $y_1 = y_2$ or $(y_1, y_2) \in E(H)$ with $x_1 = x_2\}$[5].

A graph G_1 is called as a covering graph of G with covering projection $f : G_1 \rightarrow G$ if there is a surjection $f : V(G_1) \rightarrow V(G)$ such that $f|_{N(v1)} : N(v_1) \rightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $v_1 \in f^{-1}(v)$ [6].

In this paper, we prove that the Peterson graph and the power of a cycle are b-continuous. Also, we prove that the Cartesian product of two cycles $C_m \square C_n$ is b-continuous when m and n are multiples of 5. In this case, we give the color classes of b-coloring with k colors for each k with $\chi(G) \leq k \leq b(G)$.

II. B-SPECTRUM OF C_N^K AND $C_M \square C_N$

In 2001, Lee [6] obtained the following result which helps to obtain independent dominating vertex subset of big graph from small graphs under some conditions. He proved the following theorem, which we used in the next lemma.

Theorem 2.1 [6]: Let $p : G_1 \rightarrow G$ be a covering projection and let S be a perfect dominating set of G. Then $p^{-1}(S)$ is a perfect dominating set of G_1. Moreover, if S is independent, then $p^{-1}(S)$ is independent.

In this section, we find the b-spectrum of the graphs C_n^k and $C_m \square C_n$. We prove that these two graphs are b-continues. Further, we give a method of b-coloring the graph C_n^k with i colors for each i with $\chi(G) \leq i \leq \Delta(G)+1$.

Lemma 2.2: Let $f : G \rightarrow H$ be a covering projection from a graph G on to another graph H. If the graph H is b-colorable with k colors, then so is G.

PROOF. Assume that H is b-colorable with k colors. Let the corresponding color classes be $H_1, H_2, \ldots, H_k$. Define $G_i = f^{-1}(H_i)$ for $1 \leq i \leq n$. Define $c(v) = i$ if $v \in G_i$. Since H_i's are pair wise disjoint independent vertex subsets of H, by Theorem 2.1, $\{ G_1, G_2, \ldots, G_k \}$ is a vertex partition of independent subsets of G. This means that the graph G is also k-colorable with color classes $G_1, G_2, \ldots, G_k$. It is enough to prove that each color class G_i contains a color dominating vertex.

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Let $h \in H_1$ be a color dominating vertex of H with color 1 and let $g \in f^{-1}(h) \subseteq G_1$. we prove that g is a color dominating vertex with color 1 in G.
Note that $\deg_G(g) = \deg_H(h)$. Since $h \in H_1$ is colorful, and by the definition of $\{ H_1, H_2, \ldots, H_k \}$, there exist vertices $h_2, h_3, \ldots, h_k$ such that $h_i \in H_i$ and $(h, h_i) \in E(H)$ for $2 \le i \le k$. Since $f|_{N(g)} : N(g) \to N(h)$ is a bijection, there exist vertices $g_1, g_2, \ldots, g_k \in V(G)$ such that $(g, g_i) \in E(G)$ and $f(g_i) = h_i$ for $2 \le i \le k$. Hence $g_i \in G_i$ and $c(g_i) = i$ for $2 \le i \le k$ and so g is a colorful vertex of G with color 1. Similarly, we can prove that there exist colorful vertices in G for all colors 2,3, …, k.

In [2] S. Chandra Kumar et al. proved the following Lemma.

Lemma 2.3[2]: If $k+1 \le d \le 2k+1$ and d divides n, then the graph $G = C_n^k$ admits b-coloring with d colors. In particular, when d = k+1, the fall chromatic number $\chi(G) = k+1$.

Here, we prove the above lemma more generally.

Lemma 2.4: Let $k+1 \le d \le 2k+1$. Then the graph $G = C_n^k$ admits b-coloring with d colors.

PROOF. Let $V(G) = \{ (v_n=)v_0, v_1, \ldots, v_{n-1} \}$ and $E(G) = E^1 \cup E^2 \cup \ldots \cup E^k$, where $E^i = \{ (v_j, v_{(j+i)(\text{mod } n)}) : 0 \le j \le n-1 \}$ Let n = id+t for some t with $1 \le t \le 2k$.
Case 1: If $1 \le t \le k + 1$.
Let us color the vertices as follows: For each j with $0 \le j \le id$, color of the vertex v_j is defined by $c(v_j) = j(\text{mod } d)$. Also $c(v_{id+1}) = k+1$, $c(v_{id+2}) = k+2, \ldots, c(v_{id+t}) = k+t$.
Case 2: If $k + 2 \le t \le 2k+1$.
Let us color the vertices as follows: For each j with $0 \le j \le n-1$, color of the vertex v_j is defined by $c(v_j) = j(\text{mod } d)$.
Note that, for each g with $1 \le g \le k$, the vertex v_j has exactly two neighbors $v_{j\oplus g}$ and $v_{j\oplus(n-g)}$, where $\oplus$ is the operation, addition modulo n. Hence $N(v_j) = \{ v_{j\oplus 1}, v_{j\oplus 2}, \ldots, v_{j\oplus k}, v_{j\oplus(n-1)}, v_{j\oplus(n-2)}, \ldots, v_{j\oplus(n-k)} \}$. Note that, two vertices v_a and v_b receive the same color only when $a \oplus_d b \le k+1 \le d$, where $\oplus_d$ is the operation, addition modulo d. Hence the adjacent vertices will receive different colors. In cases 1 and 2, the vertices $v_1, v_2, \ldots, v_g$ are colorful vertices with colors 1, 2, …, g respectively.

Theorem 2.5 : $S_b(C_n^k) = [\chi(C_n^k), b(C_n^k)]$.

PROOF. Since $b(G) \le \Delta(G)+1$, we have $b(C_n^k) \le 2k+1$. By the definition of C_n^k, it contains a set $\{ v_0, v_1, v_2, \ldots, v_k \}$ of mutually pair wise adjacent vertices. Hence $\chi(C_n^k) \ge k+1$. By Lemma 2.4, it follows that $S_b(C_n^k) = [\chi(C_n^k), b(C_n^k)]$.

Remark 2.6: The graph $C_5 \square C_5$ is b-colorable with i colors for each i = 3, 4 and 5. The graph $C_5 \square C_5$ is given below:

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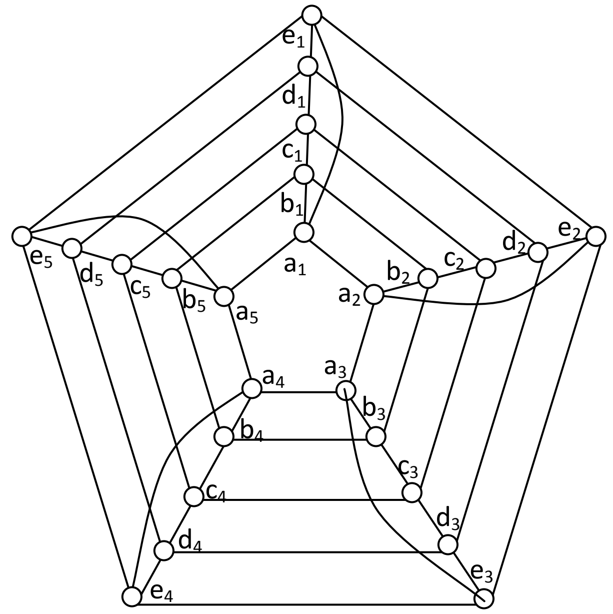

Figure 2.1: $c_5 \square c_5$

Consider the following b-coloring of $C_5 \square C_5$ with 3 colors: $c(a_1) = 1$, $c(a_2) = 2$, $c(a_3) = 1$, $c(a_4) = 3$, $c(a_5) = 2$, $c(b_1) = 2$, $c(b_2) = 3$, $c(b_3) = 2$, $c(b_4) = 1$, $c(b_5) = 3$, $c(c_1) = 1$, $c(c_2) = 2$, $c(c_3) = 1$, $c(c_4) = 3$, $c(c_5) = 2$, $c(d_1) = 3$, $c(d_2) = 1$, $c(d_3) = 3$, $c(d_4) = 2$, $c(d_5) = 1$, $c(e_1) = 2$, $c(e_2) = 3$, $c(e_3) = 2$, $c(e_4) = 1$, $c(e_5) = 3$. Note that c_1, b_1 and d_1 are colorful vertices with colors 1, 2 and 3 respectively.
Consider the following b-coloring of $C_5 \square C_5$ with 4 colors: $c(a_1) = 3$, $c(a_2) = 2$, $c(a_3) = 3$, $c(a_4) = 4$, $c(a_5) = 1$, $c(b_1) = 4$, $c(b_2) = 1$, $c(b_3) = 2$, $c(b_4) = 3$, $c(b_5) = 2$, $c(c_1) = 1$, $c(c_2) = 2$, $c(c_3) = 4$, $c(c_4) = 1$, $c(c_5) = 3$, $c(d_1) = 2$, $c(d_2) = 4$, $c(d_3) = 3$, $c(d_4) = 2$, $c(d_5) = 4$, $c(e_1) = 1$, $c(e_2) = 3$, $c(e_3) = 1$, $c(e_4) = 3$, $c(e_5) = 2$. Here a_5, b_5, c_5 and b_1 are colorful vertices with colors 1, 2, 3 and 4 respectively.
Consider the following b-coloring of $C_5 \square C_5$ with 5 colors: $c(a_1) = 4$, $c(a_2) = 2$, $c(a_3) = 5$, $c(a_4) = 3$, $c(a_5) = 1$, $c(b_1) = 5$, $c(b_2) = 3$, $c(b_3) = 1$, $c(b_4) = 4$, $c(b_5) = 2$, $c(c_1) = 1$, $c(c_2) = 4$, $c(c_3) = 2$, $c(c_4) = 5$, $c(c_5) = 3$, $c(d_1) = 2$, $c(d_2) = 5$, $c(d_3) = 3$, $c(d_4) = 1$, $c(d_5) = 4$, $c(e_1) = 3$, $c(e_2) = 1$, $c(e_3) = 4$, $c(e_4) = 2$, $c(e_5) = 5$. In this case, all the vertices are colorful vertices.

Theorem 2.7: The graph $C_m \square C_n$ is b-continuous when m and n are multiples of 5.

PROOF. Let m and n be positive integers which are multiples of 5. Since $b(G) \le \Delta(G)+1$, we have $b(C_m \square C_n) \le 5$.
If m and n are even numbers, then the graph is a product of two even cycles which is a bipartite graph and hence it has a b-coloring with 2 colors.
Otherwise, $C_m \square C_n$ contains an odd cycle and hence $\chi(C_m \square C_n) \ge 3$. From the above fact and from Remark 2.6, it follows that $S_b(C_5 \square C_5) = [\chi(C_5 \square C_5), b(C_5 \square C_5)]$.
As in the proof of Lemma0.10 in [2], there exists a covering projection from $C_m \square C_n$ to $C_5 \square C_5$. Hence the result follows form Lemma 2.2.

III. B-SPECTRUM OF PETERSON GRAPH

In this section, we find the b-spectrum of Peterson graph and we prove that it is b-continues. Throughout this section, the vertices of the Peterson graph are labeled as in the following figure. We say that the vertices a_1, a_2, a_3, a_4, a_5 are outer vertices and b_1, b_2, b_3, b_4, b_5 are inner vertices.

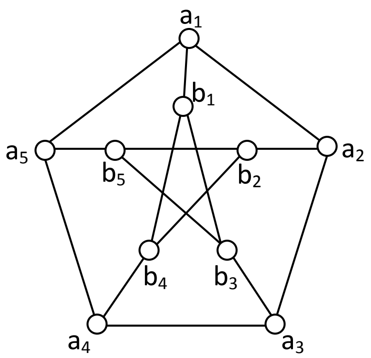

Figure 3.1: Peterson Graph

Remark 3.1: The Peterson graph P is b-colorable with 3 colors as shown in the following figure. The colorful vertices are marked by dark circles.

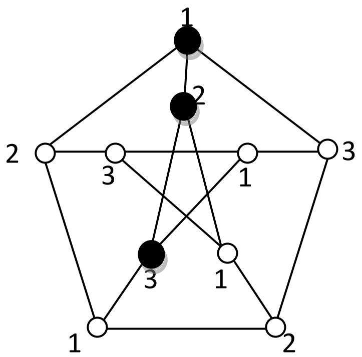

Figure 3.2

Remark 3.2: In Peterson graph, when using 4 colors, all the neighbors of a colorful vertex will receive different colors.

Lemma 3.3: The Peterson graph P is not b-colorable with 4 colors.

PROOF. Case 1: If two adjacent outer vertices of P are colorful(as shown in Fig.3.3).

Since the vertex a_2 is colorful with color 2, we should color the vertices a_3 and b_2 with colors 3 and 4.

Sub case 1.1: $c(a_3) = 4$ and $c(b_2) = 3$(as shown in Fig.3.4). Note that the adjacent vertices b_3 and b_5 cannot be colored with colors 3 and 4.

Sub case 1.1.1: $c(b_3) = 2$ and $c(b_5) = 1$(as shown in Fig.3.4.1). By Remark 3.2, the vertices a_3, a_4, a_5 and b_4 could not be colorful with color 4 and hence there exist no colorful vertex with color 4.

Sub case 1.1.2: $c(b_3) = 1$ and $c(b_5) = 2$(as shown in Fig.3.4.2). In this case, by Remark 3.2, there exists no colorful vertex with color 3.

Sub case 1.2: $c(a_3) = 3$ and $c(b_2) = 4$(as shown in Fig.3.5). Then the adjacent vertices a_4 and b_4 cannot be colored with colors 3 and 4.

Sub case 1.2.1: $c(a_4) = 1$ and $c(b_4) = 2$(as shown in Fig.3.5.1). By Remark 3.2, the vertices a_5, b_2, b_3 and b_5 could not be colorful vertices with color 4 and hence there exist no colorful vertex with color 4.

Sub case 1.2.2: $c(a_4) = 2$ and $c(b_4) = 1$(as shown in Fig.3.5.2). In this case, there exist no colorful vertex with color 3.

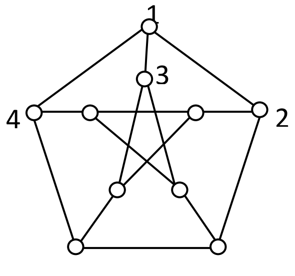

Figure 3.3

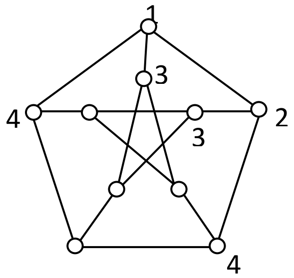

Figure 3.4

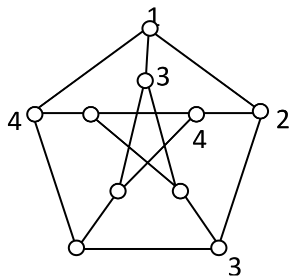

Figure 3.5

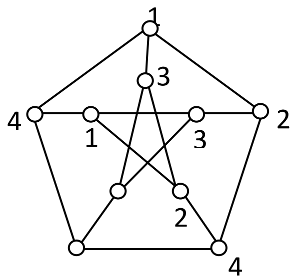

Figure 3.4.1

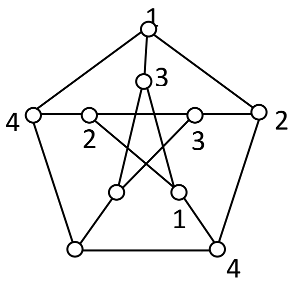

Figure 3.4.2

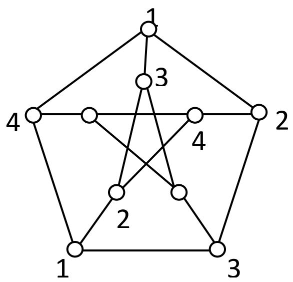

Figure 3.5.1

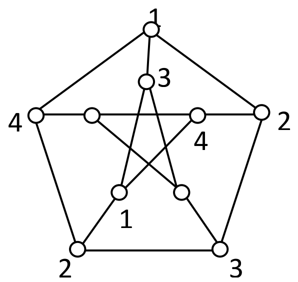

Figure 3.5.2

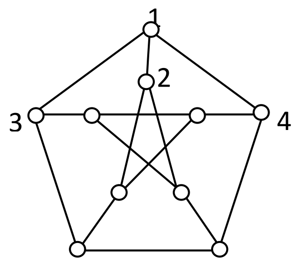

Figure 3.6

Similarly, it is not possible to have two adjacent colorful inner vertices.

Case 2: If two adjacent vertices of P (one is inner and another one is outer vertex) are colorful(as shown in Fig.3.6). Since the vertex b_1 is colorful with color 2, we should color the vertices b_3 and b_4 with colors 3 and 4.

Sub case 2.1: $c(b_3) = 4$ and $c(b_4) = 3$(as shown in Fig.3.7). Here, the adjacent vertices b_2 and b_5 cannot be colored with colors 3 and 4.

Sub case 2.1.1: $c(b_2) = 1$ and $c(b_5) = 2$(as shown in Figure 3.7.1). By Remark 3.2, the vertices a_2, a_3, a_4 and b_3 could not be colorful vertices with color 4 and hence there exist no colorful vertex with color 4.

Sub case 2.1.2: $c(b_2) = 2$ and $c(b_5) = 1$(as shown in Figure 3.7.2). In this case, by Remark 3.2, there exist no colorful vertex with color 3.

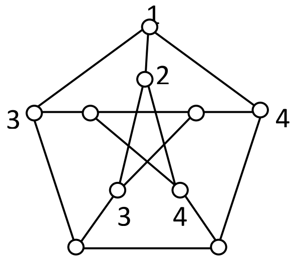

Figure 3.7

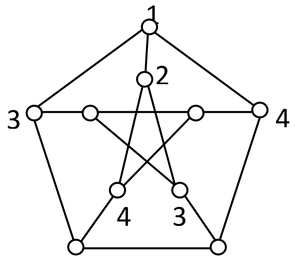

Figure 3.8

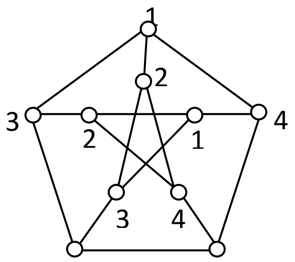

Figure 3.7.1

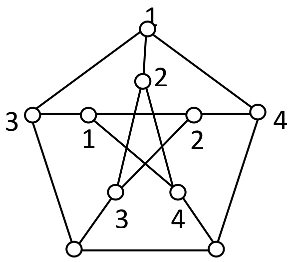

Figure 3.7.2

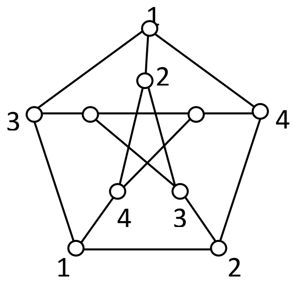

Figure 3.8.1

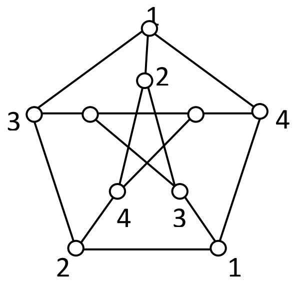

Figure 3.8.2

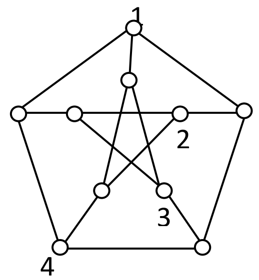

Figure 3.9

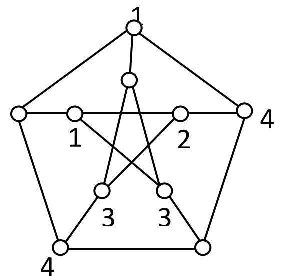

Figure 3.10

Sub case 2.2: $c(b_3) = 3$ and $c(b_4) = 4$(as shown in Fig.3.8). Then the adjacent vertices a_3 and a_4 cannot be colored with colors 3 and 4.

Sub case 2.2.1: $c(a_4) = 1$ and $c(a_3) = 2$(as shown in Fig.3.8.1). By Remark 3.2, the vertices a_2, b_2, b_4 and b_5 could not be colorful with color 4 and hence there exists no colorful vertex with color 4.

Sub case 2.2.2: $c(a_4) = 2$ and $c(a_3) = 1$(as shown in Fig.3.8.2). In this case also, there exists no colorful vertex with color 4.

From the above two cases, it is observed that the four colorful vertices must be independent and exactly two inner(outer) vertices are colorful. This is discussed in the following case.

Case 3: The Four colorful vertices are as given in Fig.3.9. The vertex b_5 may have one of the color 1 or 4 and without loss of generality, assume that $c(b_5) = 1$(as shown in Fig.3.10). Then for the colorful vertex b_2 with $c(b_2) = 2$, we should have $c(b_4) = 3$ and hence $c(a_2) = 4$(as shown in Fig.3.10). Consider the colorful vertex a_1 with $c(a_1) = 1$. Here, we should have $c(b_1) = 2$ and hence $c(a_5) = 3$(as shown in Fig.3.11). In this case, by Remark 3.2, the vertex a_4 with color 4, is not colorful.

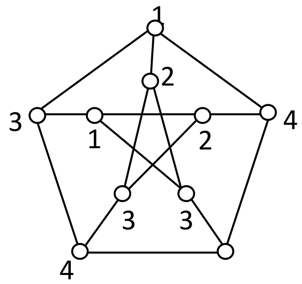

Figure 3.11

Hence, the Peterson graph is not b-colorable with 4 colors.

Since the Peterson graph is 3-regular, by Remark 3.1 and Lemma 3.3, we have the following theorem.

Theorem 3.4: The Peterson graph is b-continuous.

IV. CONCLUSION

This work may be extended in the following directions:

1. Prove that the graph $C_m \square C_n$ is b-continuous for integers m and n which are not multiples of 5.
2. Find the b-chromatic number of Cartesian product of three or more cycles.
3. Find the b-chromatic number of power of Cartesian product of two cycles and three cycles.

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