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Nucleus Equals Center in Assosymmetric Rings

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ABSTRACT: *Suppose R is a prime assosymmetric not associative ring with idempotent e and [R, R] ⊆ N then either R is associative or N = Z and e is the identity element of R if and only if e ∈ N = Z. 2010 MATHEMATICS SUBJECT CLASSIFICATION: 17A99*

KEYWORDS: *Prime ring, assosymetric ring, nucleus, center.*

I. INTRODUCTION

Kleinfeld [2] defined a class of nonassociative ring in which the associative law of multiplication has been weaked to the condition that $(P(x), P(y), P(z)) = (x, y, z)$ … (1)
for every permutation P of x, y and z. These rings are neither flexible nor power associative. But the associator and the commutator are in the nucleus of the ring. Suvarna and Jayalakshmi [3] have proved that prime assosymmetric ring is either associative or nucleus equals center. Using the results of Kleinfeld, Suvarna and Jayalakshmi we investigate the properties and the structures of prime assosymmetric rings and show that if R is prime and not assocative with an idemponent e and commutator contained in the nucleus than e is the identity element of R if and only if $e \in N = Z$. Throughout this section R represents a nonassociative assosymmetric ring. We have nucleus $N = \{n \in R \,/\, (n, R, R) = 0 = (R, n, R) = (R, R, n)\}$ and the center $C = \{c \in R \,/\, [c, R] = 0\}$. A ring is said to be prime if for any two ideals A, B such that $AB = 0$ implies either $A = 0$ or $B = 0$.

II. MAIN SECTION

Lemma 1: Suppose R has a peirce decomposition with respect to an idempotent e. Then the submodules R_{ij}, $i, j = 0, 1$ satisfy the following

(i) $R_{ii} R_{ii} \subseteq R_{ii} + R_{ji}$ $i \neq j$.
(ii) $R_{ij} R_{ji} \subseteq R_{ii} + R_{ij} + R_{ji}$, $i \neq j$.
(iii) $R_{ij} R_{ij} \subseteq R_{ij}$, $i \neq j$.
(iv) $R_{ii} R_{jj} \subseteq R_{ii} + R_{jj}$, $i \neq j$.

Proof: To prove (i) let $x_{11}, y_{11} \in R_{11}$ then using $(ex_{11} = x_{11}e = x_{11})$ we have $e(x_{11} y_{11}) = -(e, x_{11}y_{11}) + (e x_{11})y_{11} = -(y_{11}, x_{11}, e) + x_{11}y_{11} = -(y_{11} x_{11})e + y_{11} x_{11} + x_{11} y_{11}$.
Thus $e(x_{11} y_{11}) + (y_{11} x_{11})e = x_{11} y_{11} + y_{11} x_{11}$. … (2)
Assume that $x_{11} y_{11} = r_{11} + r_{10} + r_{01} + r_{00}$, $y_{11} x_{11} = s_{11} + s_{10} + s_{01} + s_{00}$ for $r_{ij}, s_{ij} \in R_{ij}$, then the identity (2) given $r_{11} + r_{10} + s_{11} + s_{10} = r_{11} + r_{10} + r_{01} + r_{00} + s_{11} + s_{10} + s_{01} + s_{00}$. Therefore $r_{00} = -s_{00}$, $r_{01} = 0 = s_{10}$ $(R_{ij} = 0 = R_{ji})$.Similarly by symmetry we obtain $r_{10} = 0 = s_{01}$ that is $x_{11} y_{11} = r_{11} + r_{00}$, $y_{11} x_{11} = s_{11} - r_{00}$. Hence $r_{11}r_{11} \subseteq r_{11} + r_{00}$.We see that $r_{10} + r_{01} \subseteq r_{11} + r_{10} + r_{01}$ if $x_{10} \in r_{10}, y_{01} \in r_{01}$.
To prove (ii) and hence $e(x_{10} y_{01}) = -(e, x_{10}, y_{01}) + (e, x_{10}) y_{01} = -(y_{01}, x_{10}, e) + x_{10}y_{01} = -(y_{01}x_{10})e + y_{01}x_{10} + x_{10}y_{01}$.Thus $e(x_{10}y_{01}) + (y_{01}x_{10})e = y_{01}x_{10} + x_{10}y_{01}$.That is $e(x_{10}y_{01}) + (y_{01}x_{10})e = x_{10}y_{01} + y_{01}x_{10}$.Thus $x_{10}y_{01} = r_{11} + r_{10} + r_{01} + r_{00}$ we have $y_{01}x_{10} = s_{11} + s_{10} + s_{01} + s_{00}$ for $R_{ij}, s_{ij}, \in R_{ij}$. Hence again from (2) we get $r_{11} + r_{10} + s_{11} + s_{10} = r_{11} + r_{10} + r_{01} + r_{00} + s_{11} + s_{10} + s_{01} + s_{00}$. Therefore $r_{11} = -s_{11}$, $r_{00} = 0 = s_{00}$. Now consider $x_{10} y_{01} = r_{11} + r_{10} + r_{01}$ and $y_{01}x_{10} = -r_{11} + s_{10} + s_{01}$.We see that $R_{10}R_{01} \subseteq R_{11} + R_{10} + R_{01}$.
Consider $x_{10} \in R_{10}, y_{10} \in R_{10}$. Then $e(x_{10} y_{10}) = -(e, x_{10}, y_{10}) + (ex_{10})y_{10} = -(y_{10}, x_{10}, e) + x_{10}y_{10} = -(y_{10}x_{10})e + y_{10}x_{10} + x_{10}y_{10}$. Also $e(x_{10}y_{01}) + (y_{10}x_{10})e = y_{10}x_{10} + x_{10}y_{10}$, $e(x_{10}y_{10}) + (y_{10}x_{10})e = x_{10}y_{10} + y_{10}x_{10}$. We see that$x_{10}y_{10} = r_{11} + r_{10} + r_{01} + r_{00}$ and $y_{10}x_{10} = s_{11} + s_{10} + s_{01} + s_{00}$ for $r_{ij}, s_{ij} \in R_{ij}$. Hence from (2) $r_{11} + r_{10} + s_{11} + s_{01} = r_{11} + r_{10} + r_{01} + r_{00} + s_{11} + s_{10} + s_{01} + s_{00}$ and $r_{10} = -s_{01}, r_{11} = 0 = s_{11}$, $r_{00} = 0 = s_{00}$. But $x_{10}y_{10} = r_{01} + r_{00}y_{10} x_{10} = s_{01} - r_{10}$. Hence obtain $R_{10}R_{10} \subseteq R_{10} + R_{01}$ i.e, $R_{ij}R_{ij} \subseteq R_{ij} + R_{ji}$.
To prove (iv) considering $R_{11}R_{00} = R_{00}R_{11} \subseteq R_{11} + R_{00}$ and $x_{11} \in R_{11}, y_{00} \in R_{00}$ we have
$e(x_{11}y_{00}) = -(e, x_{11}, y_{00}) + (ex_{11})y_{00} = -(y_{00}, x_{11}, e) + x_{11}y_{00} = -(y_{00}x_{11})e + y_{00}x_{11} + x_{11}y_{00}$. Hence $e(x_{11}y_{00}) + (y_{00}x_{11})e = y_{00}x_{11} + x_{11}y_{00}$. That is $e(x_{11}y_{00}) + (y_{00}x_{11})e = x_{11}y_{00} + y_{00}x_{11}$. So $x_{11}y_{00} = r_{11} + r_{10} + r_{01} + r_{00}$ and $y_{00}x_{11} = s_{11} + s_{10} + s_{01} + s_{00}$ for $r_{ij}, s_{ij}, \in R_{ij}$. Also from (2) we see that $r_{11} + r_{10} + s_{11} + s_{01} = r_{11} + r_{10} + r_{01} + r_{00} + s_{11} + s_{10} + s_{01} + s_{00}$. Therefore $r_{00} = -s_{00}$, $r_{01} = 0 = s_{10}$, $r_{10} = 0 = s_{01}$. Thus $x_{11}y_{00} = r_{11} + r_{00}$ and $y_{10}x_{11} = s_{00} - r_{00}$. Hence we have $R_{11}R_{00} \subseteq R_{11} + R_{00}$. ♦

Lemma 2: Suppose R has an idempotent e such that $(e, e, R) = (0) = (e, R, e)$. If R has the property $[R, R] \subseteq N$ then R has a peirce decomposition and R_{ij} satisfy

(i) $R_{ij} R_{ij} \subseteq R_{ii} + R_{jj}$ $i \neq j$
(ii) $R_{ii} R_{ji} = (0) = R_{ij}R_{ii}$ $i \neq j$
(iii) $R_{ij} R_{ij} \subseteq R_{ij}$ $i \neq j$
(iv) $R_{ij} R_{jj} \subseteq R_{ii} + R_{jj}$ $i \neq j$

Proof: To prove (i) let $x_{11} \in R_{11}$, $x_{00} \in R_{00}$

Since $x_{11} = [e, x_{11}] \in N$ and $x_{00} = [x_{00}, e] \in N$ we have $0 = (x_{11}, y_{00}, e) = (x_{11}y_{00})e - x_{11}y_{00}$ and $0 = (e, y_{00}, x_{11}) = -e\,(y_{00}x_{11})$. Or $(x_{11}y_{00})e = x_{11}y_{00}$ and $e(y_{00}x_{11}) = 0$. Similarly $e(x_{11}y_{00}) = x_{11}y_{00}$ and $(y_{00}x_{11})e = 0$. But $e(x_{11}y_{00}) = -(e, x_{11}, y_{00}) + (ex_{11})y_{00} = -(y_{00}, x_{11}e) + x_{11}y_{00} = -(y_{00}x_{11})e + y_{00}x_{11} + x_{11}y_{00}$. Hence $e(x_{11}y_{00}) + (y_{00}x_{11})e = y_{00}x_{11} + x_{11}y_{00}$. That is $x_{11}y_{00} + 0 = x_{11}y_{00} + y_{00}x_{11}$. There fore we obtain $y_{00}x_{11} = 0$. Hence we see that $R_{11}R_{11} \subseteq R_{11} + R_{00}$

To prove (ii) consider $R_{11}R_{01} = (0) = R_{10}R_{11}$. Let $x_{11} \in R_{11}$ and $x_{01} \in R_{01}$. Now $x_{11} = [e, x_{11}] \in N$ and $x_{01} = [x_{01}, e] \in N$. Thus $0 = (x_{11}, y_{01}, e) = (x_{11}y_{01})e - x_{11}y_{01}$ and $0 = (e, y_{01}, x_{11}) = -e(y_{01}x_{11})$. Or $(x_{11}y_{01})e = x_{11}y_{01}$ and $e(y_{01}x_{11}) = 0$. Similarly $e(x_{11}y_{01}) = x_{11}y_{01}$ and $(y_{01}x_{11})e = 0$. That is $e(x_{11}y_{01}) = -(e, x_{11}, y_{01}) + (ex_{11})y_{01} = -(y_{01}, x_{11}e) + x_{11}y_{01} = -(y_{01}x_{11})e + y_{01}x_{11} + x_{11}y_{01}$. Thus $e(x_{11}y_{01}) = 0 + y_{01}x_{11} + x_{11}y_{01}$ which is nothing but $x_{11}y_{01} = y_{01}x_{11} + x_{11}y_{01}$

$x_{11}y_{01} = 0$. That is $R_{11}R_{01} = 0$. ...(3)

Now let $x_{10} \in R_{10}$, $x_{11} \in R_{11}$ and $x_{10} = [e, x_{10}] \in N$ and $x_{11} = [x_{11}, e] \in N$. Then we have $0 = (x_{10}, y_{11}, e) = (x_{10}y_{11})e - x_{10}y_{11} = (e, y_{11}, x_{10}) = -e(y_{11}x_{10})$. Or $(x_{10}y_{11})e = x_{10}y_{11}$ and $e(y_{11}x_{10}) = 0$. Similarly $e(x_{10}y_{11}) = x_{10}y_{11}$ and $(y_{11}x_{10})e = 0$.That is $e(x_{10}y_{11}) = -(e, x_{10}, y_{11}) + (ex_{10})y_{11} = -(y_{11}, x_{10}, e) + x_{10}y_{11} = -(y_{11}x_{10})e + y_{11}x_{10} + x_{10}y_{11}$. Hence $x_{10}y_{11} = -0 + y_{11}x_{10} + x_{10}y_{11}$ that is $y_{11}x_{10} = 0$ which is nothing but $R_{11}R_{10} = (0)$ … (4)

From (3) & (4) $R_{11}R_{01} = (0) = R_{11}R_{10}$. That is $R_{ii}R_{ji} = (0) = R_{ii}R_{ij}$.

To prove (iii) consider $R_{10}R_{10} \subseteq R_{10}$ and let $x_{10} \in R_{10}$, $x_{10} \in R_{10}$, $x_{10} = [e, x_{10}] \in N$ and $x_{10} = [x_{10}, e] \in N$. Then we have $0 = (x_{10}, y_{10}, e) = (x_{10}y_{10})e - x_{10}y_{10}$ and $0 = (e, y_{10}, x_{10}) = -e(y_{10}x_{10})$. Or $(x_{10}y_{10})e = x_{10}y_{10}$ and $e(y_{10}x_{10}) = 0$. Similarly $e(x_{10}y_{10}) = x_{10}y_{10}$ and $(y_{10}x_{10})e = 0$. So $e(x_{10}y_{11}) = -(y_{10}x_{10})e + y_{10}x_{10} + x_{10}y_{10}$.That is $x_{10}y_{10} = -0 + y_{10}x_{10} + x_{10}y_{10}$. Hence $y_{10}x_{10} = 0$. That is $R_{10}R_{10} \subseteq R_{10}$ which shows that $R_{ij} \subseteq R_{ij}$.

Now to prove (iv) we shall consider $R_{11}R_{00} \subseteq R_{11} + R_{00}$ and $x_{11} \in R_{11}$, $x_{00} \in R_{00}$.

Since $x_{11} = [ex_{11}] \in N$ and $x_{00} = [x_{00}e] \in N$ we have $0 = (x_{11}, y_{00}e) = (x_{11}y_{00})e - x_{11}y_{00}$ and $0 = (ey_{00}, x_{11}) = -e(y_{00}x_{11})$. Or $(x_{11}y_{00})\,e = x_{11}y_{00}$ and $(y_{00}x_{11})e = 0$. Similarly $e(x_{11}y_{00}) = x_{11}y_{00}$ and $(y_{00}x_{11})e = 0$. $e(x_{11}y_{00}) = -(y_{00}x_{11})e + y_{00}x_{11} + x_{11}y_{00}$. That is $x_{11}y_{00} = 0 + y_{00}x_{11} + x_{11}y_{00}$. Hence $x_{11}y_{00} = 0$ which is nothing but $R_{ii}R_{jj} \subseteq R_{ii} + R_{jj}$. ♦

Theorem 1: Let R be a prime ring with an idempotent e. If R is not associative then e is the identity element of R if and only if $e \in N$.

Proof: Assume that $e \in N$ so $e \in Z$. Consider the peirce decomposition $R = R11 + R10 + R01 + R00$ of R with respect to e. Since $R10 = eR_{10} = R_{10}e = (0)$. We have $R = R_{01}e = eR_{01}$ (0) and $R = R_{11} + R_{00}$. Also $e \in N$ implies that R_{11} and R_{00} are ideals of R. And also since R is prime, $e \in R_{11}$ implies that $R_{00} = (0)$. Thus $R = R_{11}$ and e is the identity element of R. Conversely, let e be the idempotent element, then $ex = xe = x$ and see we have $(e, x, y) = (ex)y - e(xy) = xy - xy = 0$. That is $(e, x, y) = 0$ implies $e \in N$. ♦

A careful inspection of the Lemma 2 and Theorem 1 in [3] shows that $[R, N]$ can be replaced to by the weak condition $[R, R]$ contained in the nucleus thus we have the following theorem

Theorem 2: Let R be a prime ring such that $[R, R] \subseteq N$ Then either R is associative or $N = Z$. ♦

From Lemma 1 and Lemma 2, Theorem 1 and Theorem 2 we obtain the mainTheorem

Main Theorem 3: If R is prime not associative with an idempotent e and $[R, R] \subseteq N$ then e is the identity element of R if and only if $e \in N = Z$.

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