Applications of Euler's Theorem

To real life and life science

Dr. Mangala Gurjar

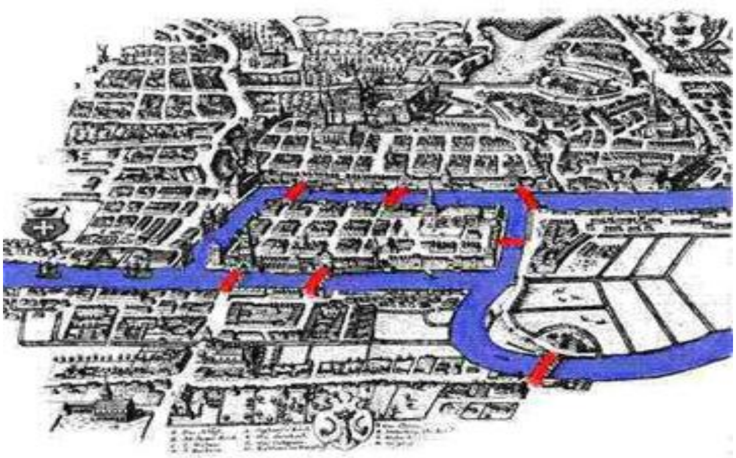
Abstract: *In the year 1735 the Swiss mathematician Euler solved the famous seven bridges problem. Euler's solution of the Königsberg's bridges problem is considered to be the first theorem of graph theory which is a branch of combinatorics. This theorem is simple yet has many applications. In the following article we discuss the seven bridges problem followed by some interesting applications of Euler's theorem.*

I. Introduction

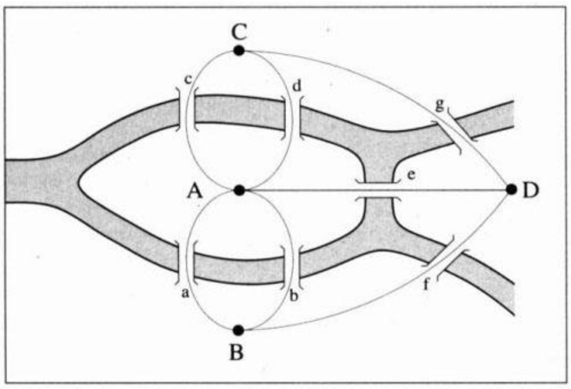
William Wordsworth described Mathematics as an independent world created out of pure intelligence. But even though the creations of Mathematicians seem to have come out of thin air by some people they have their inspirations from some concrete experience. In this paper we will see an example of a concrete problem giving rise to a mathematical theory which again has applications in real life.

Following is the aerial photograph of the town of Konigsberg in Russia. There is an island formed due to joining of two rivers. We can also see seven bridges across the rivers. For centuries people wondered whether it was possible to start at a point, walk along all the bridges exactly once and come back to the same point again. They could not do it. But that does not prove that nobody can do it. In the year 1735 Euler used schematic graph to solve the problem. He proved that it was impossible. That was the origin of Graph theory which later found applications in diverse areas. In addition Euler noticed that the key information was the number of bridges and the list of their endpoints (rather than their exact positions).

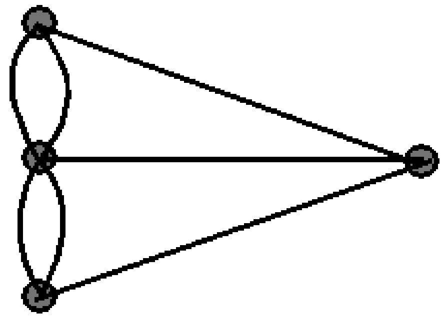
This gave rise to the development of topology:



This can be represented as:



Which can be further simplified to:



Here the vertices are the land masses shrunk to a point and lines or edges are the bridges.

Because no edge is allowed to be repeated every vertex must be entered and exited through different edges. In order to do that the degree of each vertex which is the number of lines originating at the vertex must be even. But we see that degree of each vertex is three or five which is odd. Hence it is impossible to go along all the edges exactly once and complete the circuit.

This simple observation made by Euler that a circuit can be completed including all the edges exactly once if and only if the degree of each vertex is even is called as Euler's theorem. It can be applied to many different situations. To apply Euler's theorem we have to only identify the vertices and edges. In different problems different entities become vertices and edges.

For example: Suppose you have a pack of dominos each consisting of two squares and each square containing different number of dots from the set {1, 2, 3, 4, … n}. In all there are n(n+1)/2 different dominos. Can they be arranged in a circle in such a way that squares containing same number of dots are adjacent to each other?

In order to apply graph theory to solve this problem we consider dominos as edges and numbers 1, 2, 3, etc. as vertices. All the dominos will form a complete graph where every vertex is joined to every other vertex. Thus degree of each vertex is n-1. Using Euler's theorem a circuit can be completed including all the edges exactly once if and only n-1 is even which means n is odd. In conclusion they can be arranged in a circle if and only if n is odd.

We now turn our attention to a 20th century application of Eulerian Graphs in the subject of DNA-recovery. DNA is a chain of four possible chemicals popularly called as A, C, T, G. The enzymes that break the chain after each G link are called G- fragments and those that break the chain after each G or T link are called T, C-fragments. The problem is that of recovering the original DNA- chain given the sets of it's G- fragments and T,C- fragments.

For example: The chain A C C G G A T C G T T C G T G has

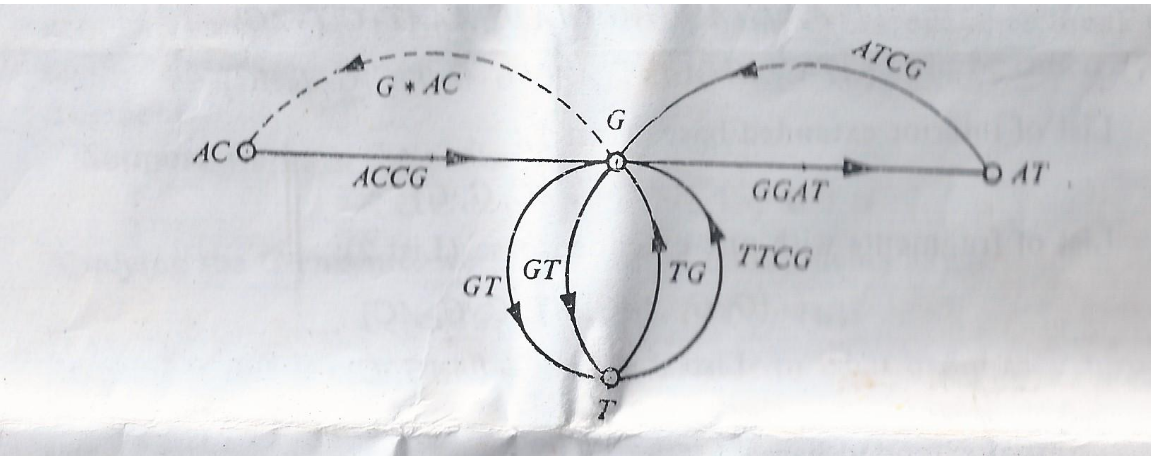
G-fragments = {ACCG, G, ATCG, TTCG, TG} and
T,C- fragments = {AC, C, GGAT, C, GT, T, C, GT, G}

Without using Euler's Theorem if we try the permutations then we have a large number of possibilities for the original sequence. But using Euler's theorem we can reduce the number of possibilities significantly as explained below.

First subject the G- fragments to T, C- fragments and subject the T, C-fragments to G- fragments thus getting the following two sets of sub fragments.
{AC.C.G, G, AT.C.G, T.T.C.G, T.G} and {AC, C, G.G. AT, C, G.T, T, C, G.T, G}
From these we collect interior sub fragments in to set-1 and fragments having only one sub fragments in to set-2. In this case,
Set-1 = {C,C, T, C, G} and Set-2 = {G, AC, C, C, T, C, G}
Set-2 will always be super set of Set-1. and Set-2 – Set-1 gives the first and the last sub fragment of the original DNA sequence. In this case Set-2 – Set-1 = {G, AC}.

Out of the two elements of this set the abnormal fragment will be the last.
Abnormal fragment is a G- fragment not ending in G or a T, C fragment not ending in T or C. In this case G is abnormal because it is a T, C fragment not ending in T or C. Thus the original chain must start with AC and end with G.

Now we build a graph with all the fragments having more than one sub fragment as the edges. In this case the set is
{ACCG, ATCG, TTCG, TG, GGAT, GT,GT}. The set of vertices will be identified as we build the graph.



In this graph there are 12 paths starting at AC and ending at G containing all the edges only once. We can add the last edge from G to AC to complete the circuit. Each of these circuits generates a sequence. One among them is the original sequence. Note that the graph constructed above has Eulerian path because the degree of each vertex is even.

Conclusion

The use of Graph Theory has reduced the number of cases to 12 from hundreds of possible cases in DNA recovery.

References

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Address Dr. Mangala R. Gurjar, M.Sc, Ph.D.(Mathematics)
Associate Professor & HoD, Mathematics department,
St. Xavier’s College, Mumbai-400001.