Some Dynamical Behaviours of a Two Dimensional Nonlinear Map

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Abstract*: We consider the Nicholson Bailey model*

f(x,y)= $(L\,x\,e^{-a\,y}, x(1-e^{-ay}))$

Where L and a are adjustable parameters, and analyse dynamical behaviours of the model. It is observed that the steady state occurs when there is no predator and prey for a certain range of the control parameters and that there exists a certain region of the control parameters in which the natural equilibrium state never occurs. In that case a modified version of the model is considered by taking care of the unboundedness of the prey system. It is further found that the model follows the stability of period-doubling fashion obeying Feigenbaum universal constant δ and at last attains infinite period doubling route leading to chaos in the system. The bifurcation points are calculated numerically and after that the accumulation point i.e. onset of chaos is calculated based on the experimental values of bifurcation points.

Key Words: *Period-Doubling Bifurcation/ Periodic orbits / Feigenbaum Universal Constant Accumulation point*
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I. Introduction:

The Nicholson Bailey model **[14]** was developed in 1930's to describe population dynamics of host-parasite (predator-prey) system. It has been assumed that parasites search hosts at random and that both parasites and hosts are assumed to be distributed in a non-contiguous ("clumped") fashion in the environment. However the modified version of the Nicholson-Bailey model has been discussed many times by many authors **[1, 2, 6, 9, 10, and 11]**.

In this present discussion in section 1.2 we verify the stability and dynamic behaviour of the model analytically and then in section 1.3 the modified form of the model has been taken which restricts the unboundedness of the model to some extent. The detailed dynamical behaviour of a particular form of its class has been studied and it has been observed that the map follows period doubling bifurcation route to chaos proving that the natural equilibrium changes its nature from periodic order to chaos .In section 1.4 numerical evaluations has been carried out to prove the geometrical behaviour. Lastly, in section 1.5 the calculation of the accumulation point from where chaos starts has been evaluated numerically, [3, 5, 6,8,12, and 13].

1.1 Nicholson-Bailey model:

The model as discussed by Nicholson and Bailey is as follows:

$x_{n+1} = Lx_n e^{-ay_n}$

$y_{n+1} = x_n\,(1-e^{-ay_n})$, where x_{n+1} represents the number of hosts (or prey) at stage n and y_{n+1} represents number of parasites(or predator) at n th stage. The difference equation can also be written in the function form as follows:

f(x,y)= $(L\,x\,e^{-a\,y}, x(1-e^{-ay}))$

1.2.1 Steady state of the above system:

The fixed point is given as follows:

$L\,x\,e^{-a\,y} = x$ **(1.2.1.1)**

$x(1-e^{-ay}) = y$ **(1.2.1.2)**

Clearly (0,0) is one of the fixed points . Let $x \neq 0$ then

$e^{-ay} = \frac{1}{L}$ $i.e. -ay = \log\left(\frac{1}{L}\right)$ $i.e. y = -\frac{1}{a}\log(\frac{1}{L})$ from (1.2.1.1)

From (1.2.1.2) we have

$x\left(1-\frac{1}{L}\right) = -\frac{1}{a}\log(\frac{1}{L})$

i.e $x = -\frac{\frac{1}{a}\log\left(\frac{1}{L}\right)}{1-\frac{1}{L}}$

Thus the fixed points are $\left(-\frac{\frac{1}{a}\log\left(\frac{1}{L}\right)}{1-\frac{1}{L}}, -\frac{1}{a}\log(\frac{1}{L})\right)$ and (0,0). However at L=1,(1.2.1.1) gives y=0 and it automatically satisfy (1.2.1.2) for any value of x. Hence any (x,0) is a fixed point for L=1.

1.3.2 Stability of the equilibrium points:

Now the Jacobian matrix is given by

$\begin{pmatrix} Le^{-ay} & -aLxe^{-ay} \\ 1-e^{-ay} & axe^{-ay} \end{pmatrix}$ The eigenvalues of which are:

$$\frac{1}{2}e^{-ay}(L+ax—4ae^{ay}Lx+(L+ax)^2 \ \ and$$

$\frac{1}{2}e^{-ay}\left(L+ax-\sqrt{-4ae^{ay}Lx+(L+ax)^2}\right)$

For fixed point (0,0) , the eigenvalues are 0, L. This shows that (0,0) is a stable solution till L=1. However for other fixed points say (x,y) , we have

$e^{-ay} = \frac{1}{L}$, hence the eigenvalues become

$$\frac{1}{2L}\left(L+ax-\sqrt{-4aL^2x+(L+ax)^2}\right) \ \ and \ \ \frac{1}{2L}\left(L+ax+\sqrt{-4aL^2x+(L+ax)^2}\right)$$

In particular for L=1, the eigenvalues are ax, 1. Thus if ax<1 one of the eigenvalues become less than 1.That is why at L=1 the trajectory converges to (x,0) such that ax<1.

Now for the period- doubling bifurcation point,

$\frac{1}{2L}\left(L+ax+\sqrt{-4aL^2x+(L+ax)^2}\right)=-1$

i.e. $L+ax+\sqrt{-4aL^2x+(L+ax)^2}$=-2L

i.e. $3L+ax=\sqrt{-4aL^2x+(L+ax)^2}$

i.e. $(3L+ax)^2=-4aL^2x+(L+ax)^2$

i.e.$8L^2+4aLx+4aL^2x=0$

i.e.$2L+ax+aLx=0$

i.e. L=$-\frac{ax}{2+ax}$ **(1.2.2.1)**

Putting x= $-\frac{\frac{1}{a}\log\left(\frac{1}{L}\right)}{1-\frac{1}{L}}$ in (1.2.2.1) we have $ax=\frac{\log(t)}{t-1}$

From eq (1.2.2.1) we have,

L=$-\frac{\log(t)}{(t-1)(2+\frac{\log(t)}{t-1})}$

i.e. $\frac{1}{t}=\frac{-\log(t)}{2(t-1)+\log(t)}$ i.e. $-tlog(t)-\log(t)=2(t-1)$

i.e.$-logt(t+1)=2(t-1)$

if t>1 then l.h.s. is negative and r.h.s. is positive.

If t<1 then l.h.s. is positive but r.h.s. is negative. Hence there is no solution i.e. period doubling bifurcation does not occur.

$\frac{1}{2L}\left(L+ax+\sqrt{-4aL^2x+(L+ax)^2}\right)=1$

i.e. $L+ax+\sqrt{-4aL^2x+(L+ax)^2}$=2L

i.e. $-L+ax=\sqrt{-4aL^2x+(L+ax)^2}$

i.e. $(-L+ax)^2=-4aL^2x+(L+ax)^2$

i.e.$-4aLx+4aL^2x=0$

i.e.$-ax+aLx=0$

i.e. L=1 **(1.2.2.2)**

Again we consider the stability of the other fixed point $\left(-\frac{\frac{1}{a}\log\left(\frac{1}{L}\right)}{1-\frac{1}{L}},-\frac{1}{a}\log\left(\frac{1}{L}\right)\right)$

Now we consider the expression $-4aL^2x+(L+ax)^2$ for the fixed point $\left(-\frac{\frac{1}{a}\log\left(\frac{1}{L}\right)}{1-\frac{1}{L}},-\frac{1}{a}\log\left(\frac{1}{L}\right)\right)$.The simplified expression is

$\frac{-2(-1+L)^2(-1+2L)+2\mathrm{Log}[L](-1+L^2-L\mathrm{Log}[L])}{(-1+L)^3L}$=g(L)

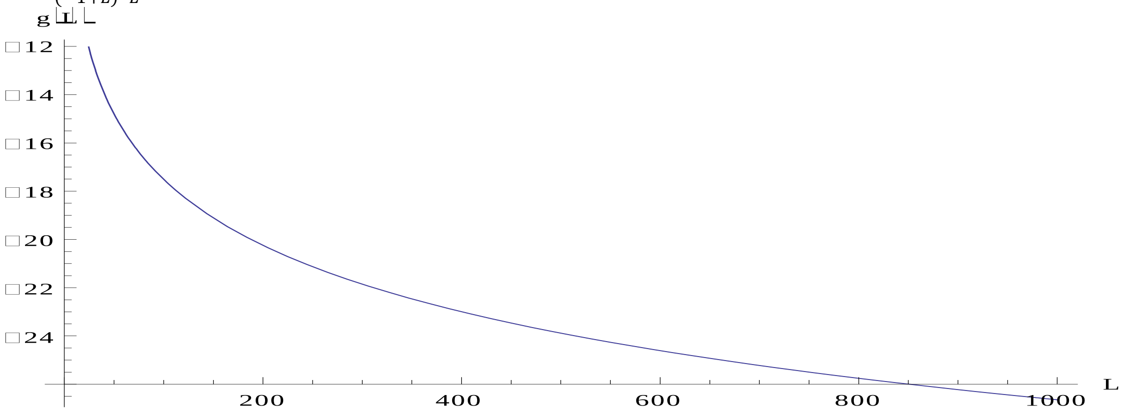

Fig 1.2.2.a : Abcissa represents the control parameter L and ordinate represents g(L)

Clearly,g(L) is negative for L>1

Hence magnitude of the eigenvalues become

$$\left|\frac{1}{2L}\left(L + ax - \sqrt{-4aL^2x + (L+ax)^2}\right)\right| = \frac{1}{2L}\sqrt{(L+ax)^2 - (L+ax)^2 + 4aL^2x}$$

$=\sqrt{ax} = \sqrt{\frac{LlogL}{L-1}}$=h(L)(say).

For L>1 and for large value of L, the above expression shows the graph as

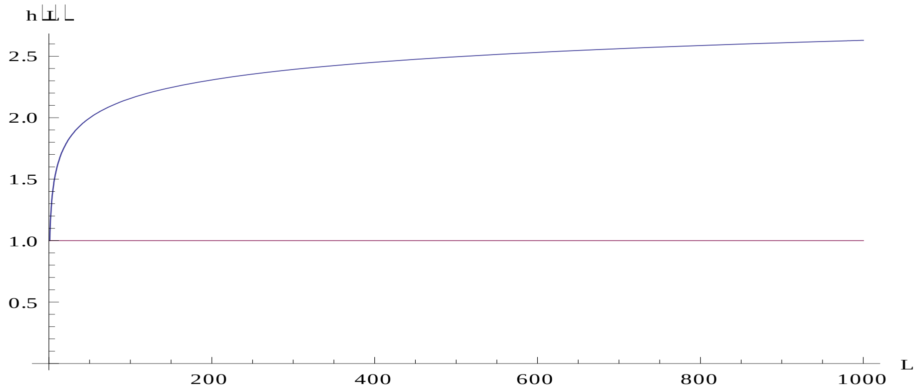

Fig 1.2.2.b: Abcissa represents the control parameter L and ordinate represents h(L)

Hence the fixed pint is unstable, and this shows that the model has been made to fulfill the fact that equilibrium stage never occurs for predator system in nature.

We now take a modified version of the Nicolson Bailey model, i.e. we take

$x_{n+1} = Lx_n e^{-ay_n - x_n^2}$

$y_{n+1} = x_n(1 - e^{-ay_n})$

The additional term $e^{(-x_n^2)}$ with x_{n+1} helps to restrict the unlimited growing of host(or prey).

1.3 Dynamical behaviour of the map keeping "a" constant:

We now fix the parameters say "a" and keep varying L to analyse the detailed dynamical behaviours of the map. Let us take a=0.1. On inspection it can be seen that (0,0) is a fixed point of the model satisfying the equation

f(x,y)=(x,y) $=(Lxe^{-ay-x^2},\ x(1-e^{-ay}))$

i.e. x= Lxe^{-ay-x^2}

y= $x(1-e^{-ay})$ **(1.3.1)**

Using "Mathematica" software we generate the bifurcation diagram for the observation of the whole dynamical behaviour of the map as L is varied.

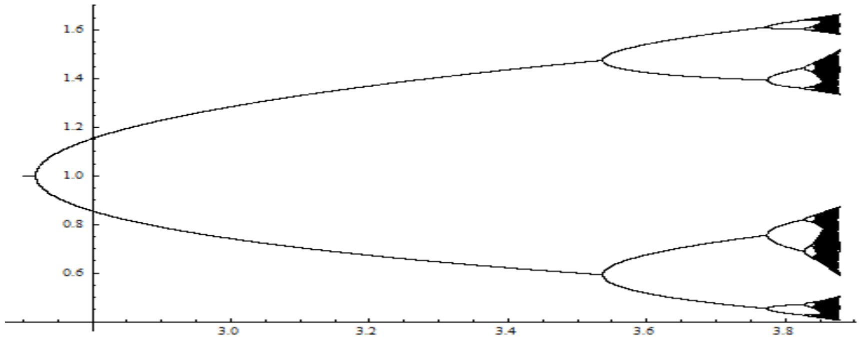

Fig 1.3.a: The figure is generated using 20000 points of which the last 300 points are taken at every parameter value of L, and plotted the x coordinate of the point (x,y) vs. L.

The eigen values of the linearised form are as follows:

$\frac{1}{2}e^{-x^2-2ay}\left(e^{ay}L + ae^{x^2+ay}x - 2e^{ay}Lx^2 \pm \sqrt{(-e^{ay}L - ae^{x^2+ay}x + 2e^{ay}Lx^2)^2 - 4e^{x^2+2ay}(ae^{ay}Lx - 2aLx^3)}\right)$, which can be re- written as

$\frac{1}{2L}\left(e^{ay}L + aLx - 2e^{ay}Lx^2 \pm \sqrt{(-e^{ay}L - aLx + 2e^{ay}Lx^2)^2 - 4e^{ay}L(ae^{ay}Lx - 2aLx^3)}\right)$

The diagram shows that the model follows period doubling route to chaos on increasing the control parameter L. For (0,0) the eigenvalues are 0, L, which says that (0,0) loses stability at L=1. Let (x_0, y_0) be a fixed point of the map f where neither of x_0, y_0 are equal to zero. The fixed point is stable till both the eigen values at x_0, y_0 are less than 1 in modulus. However the first bifurcation point can be obtained from the equations (1.2.1) and min{λ_1,λ_2}=-1.If we now begin to increase the value of L exceeding the bifurcation point , the fixed point (x_0,y_0) loses its stability and there arises around it two points, say, (x_{21}(L), y_{21}(L)) and (x_{22}(L), y_{22}(L)) forming a stable periodic trajectory of period 2. On increasing the value of L one of the eigen values starts decreasing from positive values to negative and when we reach a certain value of L, we find that one of the eigenvalues of the Jacobian of f^2 becomes -1, indicating the loss of stability of the periodic trajectory of period two. Thus, the second bifurcation takes place at this value L_2 of L. We can repeat the same process, and find that the periodic trajectory of period 2^n becomes unstable and a periodic trajectory of period 2^{n+1} appears in its neighbourhood for all n=1,2,3,…. , [5,6,8,10].

1.4 Numerical Method for Obtaining Bifurcation Points:

We have used Newton-Raphson method to obtain the periodic points which has been proved to be worthy for sufficient accuracy and time saving.

The Newton Recurrence formula is

$\overline{x_{n+1}} = \overline{x_n} - Df(\overline{x_n})^{-1} f(\overline{x_n})$, where n = 0,1,2,… and $Df(\bar{x})$ is the Jacobian of the map f at the vector $\bar{x} = (x_1, x_2)$(say) . We see that this map f is equal to f^k-I in our case, where k is the appropriate period. The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool in our map one needs a number of recurrence formulae which are given below.

Let the initial point be (x_0 , y_0) and let M(x,y)= Lxe^{-ay-x^2},N(x,y)= $x(1-e^{-ay})$,

Let $A_0=\frac{\partial M}{\partial x}|_{(x_0,y_0)}$, $B_0=\frac{\partial M}{\partial y}|_{(x_0,y_0)}$, $C_0=\frac{\partial N}{\partial x}|_{(x_0,y_0)}$, $D_0=\frac{\partial N}{\partial y}|_{(x_0,y_0)}$

$$\text{and } A_k=\begin{pmatrix} \frac{\partial M}{\partial x}|_{(x_k,y_k)} & \frac{\partial M}{\partial y}|_{(x_k,y_k)} \\ \frac{\partial N}{\partial x}|_{(x_k,y_k)} & \frac{\partial N}{\partial y}|_{(x_k,y_k)} \end{pmatrix}\begin{pmatrix} A_{k-1} & B_{k-1} \\ C_{k-1} & D_{k-1} \end{pmatrix} \quad \forall k \geq 1$$

Since the fixed point of the map f is a zero of the map

F(x,y) = f(x,y)-(x,y), the Jacobian of $F^{(k)}$ is given by

$$J_k - I = \begin{pmatrix} A_k - 1 & B_k \\ C_k & D_k - 1 \end{pmatrix}. \text{ Its inverse is } (J_k - I)^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_k - 1 & -B_k \\ -C_k & A_k - 1 \end{pmatrix}$$

where $\Delta=(A_k-1)(D_k-1)-B_kC_k$, the Jacobian determinant. Therefore, Newton's method gives the following recurrence formula in order to yield a periodic point of F^k

$$x_{n+1} = x_n - \frac{(D_k - 1)(\overline{x_n} - x_n) - B_k(\overline{y_n} - y_n)}{\Delta}$$

$$y_{n+1} = y_n - \frac{(-C_k)(\overline{x_n} - x_n) + (A_k - 1)(\overline{y_n} - y_n)}{\Delta}$$

$$\textit{where} \quad F^k(\overline{x_n}) = (x_n, y_n)$$

1.4.1 Numerical Methods for Finding Bifurcation Values:

As described above for some particular value of $L=L_1$ say , the fixed point of f^k is calculated and hence the eigenvalues of J_k can be calculated at the fixed point. Let (x_1,y_1) ,(x_2,y_2),…. (x_k,y_k) be the periodic points of f at L_1. Let λ_1, λ_2 be the two eigen values of J_k at L_1 , let $I(k,L_1)= \min\{\lambda_1, \lambda_2\}$, where $n=2^k$ is the period number .Then we search two values of "L" say L_{11} and L_{22} such that $(I(k,L_{11})+1)(I(k,L_{22})+1) < 0$ .Then the existence of n^{th} bifurcation point is confirmed in between L_{11} and L_{22}. Then we may apply some of the numerical techniques viz. Bisection method or Regula Falsi method on L_{11} and L_{22} for sufficient number of iterations to get L such that I(k,L)=-1.

Our numerical results are as follows:

Table 1.4.1.a: Bifurcation points calculated with the above numerical procedure are given as follows:

Period	Bifurcation point
n=1	2.71828182845904523536028747135266249775724709369995957496696
n=2	3.53684067130120359837043484826459115405423168841443588327400
n=4	3.77415543777691650197392802578931393394103470516281183784919
n=8	3.82788212068493762703545087588566660131789524411227589313294
n=16	3.83954703996904224281164775272528029534621679916918934834796
n=32	3.84205211923591357428498949577187590314662912792561150844422
n=64	3.84258895757543669502842561849603953670354363149648649855300
n=128	3.84270394654591339735557484998431661721920300008353132976556
n=256	3.84272857434341121053856237571767662232417602736345292594595
n=512	3.84273384889445086737896598441474166356789762228260539689561
n=1024	3.842734978543189743621418854360407234045741604974429302522640
n=2048	3.84273522047942178512266708047918352493564928024817333443963

The Feigenbaum universal constant is calculated using the experimentally calculated bifurcation point using the following formula $\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$,where A_n represents n^{th} bifurcation point. The values of δ_n are as follows.

δ_1=3.44925372743685
δ_2=4.41707460112493
δ_3=4.60583409104533
δ_5=4.65650705682990
δ_6=4.66635685725543
δ_7=4.66860723509033
δ_8=4.66907243682318
δ_9=4.66917417475885
δ_{10}=4.66919570494443
δ_{11}=4.66920034814159

It may be observed that the map obeys Feigenbaum universal behaviour as the sequence $\{\delta_n\}$ converges to δ as n becomes very large.

1.5 Accumulation Point:

The accumulation point can be calculated by the formula $A_\infty=(A_2-A_1)/(\delta-1)$, where δ is Feigenbaum constant. But it has been observed that $\{\delta_n\}$ converges to δ as n→ ∞.Therefore a sequence of accumulation point $\{A_{\infty,n}\}$is made using the formula $A_{\infty,n}=(A_{n+1}-A_n)/(\delta-1)$**[8]**. From the above experimental values of bifurcation points and using δ= 4.669201609102990671853204 the sequence of values is constructed as follows:

$A_{\infty,1}$ = 3.759929759895131081 27
$A_{\infty,2}$ = 3.83883293216342125972
$A_{\infty,3}$ = 3.84252472924989889705
$A_{\infty,4}$ = 3.84272618369752118495
$A_{\infty,5}$ = 3.84273485066736624607
$A_{\infty,6}$ = 3.84273526688103365882
$A_{\infty,7}$ = 3.84273528550809836535
$A_{\infty,8}$ = 3.84273528637510680165
$A_{\infty,9}$ = 3.84273528641454422260
$A_{\infty,10}$ = 3.84273528641636195462
$A_{\infty,11}$ = 3.84273528641644509866

It may be observed that the sequence converges to the point 3.842735286416 …. After which chaotic region starts.

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