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On The Zeros of Certain Class of Polynomials

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Abstract: *Let P(z) be a polynomial of degree n with real or complex coefficients . The aim of this paper is to obtain a ring shaped region containing all the zeros of P(z). Our results not only generalize some known results but also a variety of interesting results can be deduced from them.*

I. Introduction and Statement of Results

The following beautiful result which is well-known in the theory of distribution of zeros of polynomials is due to Enestrom and Kakeya [9].

Theorem A. If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

(1) $a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0 \geq 0,$

then all the zeros of P(z) lie in $|z| \leq 1$.

In the literature ([2] ,[5] –[6], [8]-[11]) there exists some extensions and generalizations of this famous result. Aziz and Mohammad [1] provided the following generalization of Theorem A.

Theorem B. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real positive coefficients. If $t_1 \geq t_2 \geq 0$ can be found such that

(2) $a_j t_1 t_2 + a_{j-1}(t_1 - t_2) a_{j-2} \geq 0$, j=1, 2, … ;n+1, $(a_{-1} = a_{n+1} = 0)$,

then all the zeros of P(z) lie in $\lfloor z \rfloor \leq t_1$.

For $t_1 = 1, t_2 = 0$, this reduces to Enestrom-Kakeya Theorem (Theorem A).

Recently Aziz and Shah [3] have proved the following more general result which includes Theorem A as a special case.

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Theorem C. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some t > 0.

(3) $Max_{|z|=R} \lfloor t a_0 z^n + (t a_1 - a_0) z^{n-1} + \cdots + (t a_{n-} a_{n-1}) \rfloor \leq M_3$

Where R is any positive real number, then all the zeros of P(z) lie in

(4) $\lfloor z \rfloor \leq Max \left\{ \frac{M_3}{\lfloor a_n \rfloor}, \frac{1}{R} \right\}$

The aim of this paper is to apply Schwarz Lemma to prove a more general result which includes Theorems A, B and C as special cases and yields a number of other interesting results for various choices of parameters α, r, t_1 *and* t_2. In fact we start by proving the following result.

Theorem 1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, If for some real numbers t_1, t_2 with $\boldsymbol{t_1 \neq 0, t_1 > t_2 \geq 0}$

(5) $Max_{|z|=R} \left|(a_n(t_1 - t_2) + \alpha - a_{n-1}) + \sum_{j=0}^{n} \left(a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2}\right) z^{n-j+1}\right| \leq M_1$

(6) $Min_{|z|=R} \left|a_0 (t_1 - t_2) + a_1 t_1 t_2 + \beta \right| + \sum_{j=2}^{n+2} \left(a_j t_1 t_2 + a_{j-1} (t_1 - t_2) - (a_{j-2}\right) z^j \right| \leq M_2$

Where R is a positive real number, then all the zeros of P(z) lie in the ring shaped region.

(7) $Min_{\lfloor z \rfloor = R} \left(\frac{t_1 t_2 |a_0|}{|\beta| + M_2}, R \right) \leq |z| \leq Max_{\lfloor z \rfloor = R} \left(\frac{M_1 + |\alpha|}{|a_n|}, \frac{1}{R} \right)$

Taking $t_2 = 0$, we get the following generalization and refinement of Theorem C.

Corollary 1. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some t > 0.

$Max_{|z|=R} \left|(a_n t + \alpha - a_{n-1}) + \sum_{j=0}^{n} (a_{j-1} t - a_{j-2}) z^{n-j+2}\right| \leq M_1$

$Min_{|z|=R} \left|(a_0 t + \beta) + \sum_{j=0}^{n+2} (a_{j-1} t - a_{j-2}) z^j\right| \leq M_2$

Where R is a real positive number then all the zeros of P(z) lie in

$$|z| \leq Max_{|z|=R} \left(\frac{M_1 + |\alpha|}{|a_n|}, \frac{1}{R}\right)$$

In case we take t=1=R in corollary 1, we get the \following interesting result.

Corollary 2. Let P(z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n,

$$Max_{|z|=R} \{ |a_n + \alpha - a_{n-1}| + |(a_{n-1} - a_{n-2})z + \cdots + (a_1 - a_0)z^{n-1} + a_o z^n| \} \leq M,$$

then all the zeros of P(z) lie in the circle $|z| \leq \text{Max}\left(\frac{M + |\alpha|}{|a_n|}, 1\right)$

If for some α > 0, $\alpha + a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$, then ,

$$M \leq |a_n + \alpha + a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_1 - a_0| + |a_0|$$

$$= (a_n + \alpha - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + |a_1 - a_0| + a_0$$

$$= a_n + \alpha$$

Using this observation in Corollary 2, we get the following generalization of Enestrom-Kakeya Theorem.

Corollary 3. If, P(z)= $a_n z^n + a_{n-1} + z^{n-1} + \cdots + a_1 z + a_0$, is a polynomial of degree n, such that for some $\alpha \geq 0$,

$$\alpha + a_n \geq a_{-1} \geq \cdots . \geq a_0 \geq 0,$$

then all the zeros of P(z) lie in the circle

$$|z| \leq 1 + \frac{2\alpha}{a_n},$$

For $\alpha = 0$, *this reduces to Theorem A. If we take* $\alpha = a_{n-1} - a_n \geq 0$, then we get Corollary 2, of ([4] Aziz and Zarger).

Next we present the following interesting result which includes Theorem A as a special case.

Theorem 2. Let P(z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, If for some real numbers with $\boldsymbol{t_1 \neq 0, t_1 > t_2 \geq 0}$

(8) $\boldsymbol{Max_{|z|=R}\left|\sum_{j=0}^{n}\left(a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2}\right)z^{n-j}\right| \leq M_3}$

then all the zeros of P(z) lie in the region

(9) $|z| \leq r_{1,}$

Where ,

(10) $r_1 = \frac{2M_3}{\{|a_n(t_1 - t_2)\, a_{n-1}|^2 + 4|a_n|M_3\}^{1/2} - |a_n(t_1 - t_2) - a_{n-1}|}$

Taking $t_2 = 0$, we get the following result

Corollary 4. Let P(z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, If for some $t > 0$,

$$\boldsymbol{Max_{|z|=R}\left|\sum_{j=0}^{n}\left(a_{j-1}t - a_{j-2}\right)z^{n-j}\right| \leq M_3,}$$

then all the zeros of P(z) lie in the region

$$|z| \leq r_{1,}$$

Where ,

$$\text{r} = \frac{2M_3}{\{|a_n t - a_{n-1}|^2 + 4|a_n|M_3\}^{1/2} - |a_n t - a_{n-1}|}$$

Remark. Suppose polynomial P(z) = $\sum_{j=0}^{n} a_j z^j$, satisfies the conditions of

Theorem A, then it can be easily verified that from Corollary 4.

$$M_3 = a_{n-1},$$

then all the zeros of P(z) lie in

$$|z| \le \frac{2a_{n-1}}{\sqrt{(a_n - a_{n-1})^2 + 4a_n a_{n-1}} - (a_n - a_{n-1})}$$

$$= \frac{2a_{n-1}}{2a_{n-1}} = 1,$$

which is the conclusion of Enestrom-Kekaya Theorem.

Finally , we prove the following generalization of Theorem 1 of ([2], Aziz and Shah).

Theorem 3. Let P(z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, Ifα, β are complex numbers and with $t_1 \neq 0, t_2$ *are real numbers with* $t_1 \ge t_2 \ge 0$ and

(11) $Max_{|z|=R}\left|(a_n(t_1 - t_2) + \alpha - a_{n-1})z + \sum_{j=0}^{n} a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^{n-j-2}\right| \le M_4$

and,

(12) $Max_{|z|=R}\left|(a_n(t_1 - t_2) + \beta - a_{n-1})z + \sum_{j=2}^{n+2} a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2})z^j\right| \le M_5$

Where $a_{-2=}a_{-1} = 0 = a_{n+1} = a_{n+2}$ and R is any positive real number, then all the zeros of P(z) lie in the ring shaped region.

(13) $$\text{Min}\,(r_2, R) \le |z| \le \max\left(r_1, \frac{1}{R}\right)$$

Where

(14) $$r_1 = \frac{2[|\alpha|R^2|a_n(t_{1-}t_2) - a_{n-1} + \alpha| + M_4^2]}{\{(|\alpha|R^2M_4 + R^2(M_4 - |a_n|)|a_n(t_{1-}t_2) + \alpha - a_{n-1}|^2 + 4(|\alpha|R^2|a_n(t_{1-}t_2) + \alpha - a_{n-1}| + M_4^2)(|a_n|R^2M_4)\}^{1/2} -}$$

$$(|\alpha|R^2M_4 + R^2(M_4 - |a_n|)|a_n(t_{1-}t_2) + \alpha - a_{n-1}|)$$

and

(15) $$\frac{1}{r_2} = \frac{2[R^2|\beta||\ a_1t_1t_2 + a_0(t_{1-}t_2) + \beta| - M_5^2]}{R^2\{|a_1t_1t_2 + a_0(t_{1-}t_2) + \beta|(|a_0|t_1t_2 - M_5) - |\beta|M_5\} +}$$

$$\{R^4(\ |a_1t_1t_2 + a_0(t_{1-}t_2) + \beta|(|a_0|t_1t_2 - M_5))^2$$

$$+4(|a_0|(M_5t_1t_2)(R^2|\beta||a_1t_1t_2 + a_0(t_{1-}t_2) + \beta| + M_5^2)\}^{1/2}$$

Taking $\alpha = 0, \beta = 0,$ in Theorem 3, we get the following.

Corollary 5. Let P(z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, If ,$t_1 \neq 0$ *and* t_2 are real numbers with $t_1 \ge t_2 \ge 0,$

$$Max_{|z|=R}\left|\sum_{j=1}^{n+1}\left(a_j t_1 t_2 + a_{j-1}(t_{1-}t_2) - a_{j-2}\right)z^{n-j+2}\right| \le M_4,$$

$$\mathbf{Max_{|z|=R}\left|\sum_{j=1}^{n+2}\left(a_j t_1 t_2 + a_{j-1}(t_{1-}t_2) - a_{j-2}\right)z^j\right| \le M_5},$$

Where R is any positive real number . then all the zeros of P(z) lie in the ring shaped region

$$\min(r_2, R) \le |z| \le \max\left(r_1, \frac{1}{R}\right)$$

Where

$$r_1 = \frac{2M_4^2}{\{R^4|(t_{1-}t_2)a_n - a_{n-1}|^2(M_4 - |a_n|)^2 + 4|a_n|R^2M_4^3\}^{1/2} - |(t_1 - t_2)a_n - a_{n-1}|(M_4 - |a_n|)R^2}$$

$r_2 =$

$$\frac{1}{2M_5^2}\Big[\{R^4|a_1t_1t_2+a_0(t_1-t_2)|^2(M_5-|a_0|t_1t_2)^2+4M_5^3R^2|a_0|t_1t_2\}^{\frac{1}{2}}- R^2(M_5-|a_0|t_1t_2)|a_1t_1t_2+a_0(t_1-t_2)|\Big]$$

The result was also proved by Shah and Liman [12].

II. LEMMAS

For the proof of these Theorems, we need the following Lemmas. The first Lemma is due to Govil, Rahman and Schmesser [7].

LEMMA. 1. If $f(z)$ is analytic in $|z| \le 1, f(o) = a$ *where* $|a| < 1, f'(o) = b, |f(z)| \le 1,$ *on* $|z| = 1,$ *then for* $|z| \le 1.$

$$|f(z)| \le \frac{(1-|a|)|z|^2+|b||z|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|b||z|+(1-|a|)}$$

The example

$$f(z)=\frac{a+\frac{b}{1+a}z-z^2}{1-\frac{b}{1+a}z-az^2}$$

Shows that the estimate is sharp.

form Lemma 1, one can easily deduce the following:

LEMMA. 2. If $f(z)$ is analytic in $|z| \le R, f(0) = 0, f'^{(0)} = b$ *and* $|f(z)| \le M,$

for $|z| = R,$ *then*

$$|f(z)| \le \frac{M|z|}{R^2}\frac{M|z|+R^2|b|}{M+|b||z|} \text{ for } |z| \le R$$

III. PROOFS OF THE THEOREMS.

Proof of Theorem . 1 Consider

(16)

$$F(z) = (t_2+z)(t_1-z)P(z)$$

$$= -a_nz^{n+2}+(a_n(t_1-t_2)-a_{n-1})z^{n+1}$$

$$+(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2})z^n$$

$$+\ldots+(a_nt_1t_{2+}a_0(t_1-t_2))z+a_0t_1t_2$$

Let

$$G(z) + z^{n+2}F\left(\frac{1}{z}\right)$$

$$= -a_n+(a_n(t_1-t_2)-a_{n-1})z+(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2})z^2+\cdots$$

$$\ldots+(a_1t_1t_2+a_0(t_1-t_2)z^{n+1}+a_0t_1t_2)z^{n+2}$$

$$= -a_n-\alpha z+(a_n(t_1-t_2)+\alpha-a_{n-1})z$$

$$+(a_nt_1t_2+a_{n-1}(t_1-t_2)-a_{n-2})z^2+\cdots$$

$$\ldots+(a_1t_1t_2+a_0(t_1-t_2)z^{n+1}+a_0t_1t_2)z^{n+2}$$

$$= -(a_n+\alpha z)+H(z)$$

Where,

$$H(z)= z\{(a_n(t_1-t_2)+\alpha-a_{n-1})+\textstyle\sum_{j=0}^{n}(a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2})z^{n-j}\}$$

Clearly H(o)=0 and $Max_{|z|=R}\ |H(z)| \le RM_1$,

We first assume that $|a_n| \le R(M_1 + |\alpha|)$

Now for $|z| \le R$, by using Schwarz Lemma, we have

(17) $|G(z)| = |-(a_n + \alpha z) + H(z)|$

$$\ge |a_n| - |\alpha||z| - |H(z)|$$

$$\ge |a_n| - |\alpha||z| - |M_1 |z|$$

$$= |a_n| - (M_1 + |\alpha|)|z|$$

$$> o, if$$

$$|z| < \frac{|a_n|}{M_1+|\alpha|} \ (\le R)$$

This shows that all the zeros of G(z) lie in $|z| \ge \frac{|a_n|}{M_1+|\alpha|}$

Replacing z by $\frac{1}{z}$ and noting that F(z) = $z^{n+2}G\left(\frac{1}{z}\right)$, it follows that all the zeros of F(z) lie in

$$|z| \le \frac{M_1+|\alpha|}{|a_n|} \text{ if }, |a_n| \le R\ (M_1 + |\alpha|)$$

Since all the zeros of P(z) are also the zeros of F(z), we conclude that all the zeros of P(z) lie in

(18) $$|z| \le \frac{|a_n|}{M_1+|\alpha|}$$

Now assume $|a_n| > R\ (M_1 + |\alpha|)$, then for $|z| \le R$, we have from (17).

$$|G(z)| \ge |a_n| - |\alpha||z| - H(Z)$$

$$\ge |a_n| - |\alpha|R - M_1 R$$

$$= |a_n| - (|\alpha| + M)R > 0,$$

Thus G(z) $\ne 0$ $for\ |z| < R, from\ which$ it follows as before that all the zeros of F(z) and hence all the zeros of P(z) lie in $|z| \le \frac{1,}{R}$ Combining this with (18), we infer that all the zeros of P(z) lie in

19. $$|\mathbf{z}| \le \mathbf{max}\left\{\frac{|\mathbf{a_n}|}{\mathbf{M_1}+|\boldsymbol{\alpha}|}, \frac{\mathbf{1,}}{\mathbf{R}}\right\}$$

Now to prove the second part of the Theorem, from (16) we can write F(z) as

$$F(z)= (t_2 + z)(t_1 - z)P(z)$$

$$=$$

$$a_0t_1t_2 + \big(a_1t_1t_2 + a_0(t_1 - t_2)\big)z + \cdots + (\ a_n(t_1 - t_2) - a_{n-1}\ )z^{n+1} - a_n z^{n+2}$$

$$=a_0t_1t_2 - \beta z + (a_1t_1t_2 + a_0(t_1 - t_2) + \beta)z + \cdots + (\ a_n(t_1 - t_2) - a_{n-1}\ )z^{n+1} - a_n z^{n+2}$$

$$= a_0t_1t_2 - \beta z + T(z)$$

Where

$$T(z) = z\{(a_1t_1t_2 + a_0(t_1 - t_2) + \beta) + \textstyle\sum_{j=2}^{n+2} a_j t_1 t_2 + a_{j-1}(t_1 - t_2)a_{j-2}\ \}z^{j-1}$$

Clearly T(o)=0, and $Max_{|z|=R}|T(z)| \le RM_2$

We first assume that $|a_0t_1t_2| \le |\beta| + M_2$

Now for $|z| \le R$, by using Schwasr lemma, we have

$$|F(z)| \ge |a_0 t_1 t_2| - |\beta||z| - |T(z)|$$

$$\ge |a_0 t_1 t_2| - |\beta||z| - M_2|Z|$$

>0, if,

$$|z| < \frac{a_0 t_1 t_2}{|\beta| + M_2} \qquad (\le R)$$

This shows that F(z) ≠0, for $|z| < \frac{|a_0 t_1 t_2|}{|\beta| + M_2}$, hence all the zeros of F(z) lie in

$$|z| \ge \frac{|a_0 t_1 t_2|}{|\beta| + M_2}$$

But all the zeros of P(z) are also the zeros of F(z), we conclude that all the zeros of P(z) lie in

20. $$|z| \ge \frac{|a_0\ t_1\ t_2|}{|\beta| + M_2}$$

Now we assume $|a_0\ t_1\ t_2| > |\beta| + M_2$, then for $|z| \le R$, we have

` $$|F(z)| \ge |a_0 t_1 t_2| - |\beta||z| - |T(z)|$$

$$\ge |a_0 t_1 t_2| - |\beta|R - M_2 R$$

$$> |t_1 t_2 a_0| - (\ |\beta| + M_2\ )R$$

>0,

This shows that all the zeros of F(z) and hence that of P(z) lie in

$$|z| \ge R$$

Combining this with (20) , it follows that all the zeros of P(z) lie in

21 $$|z| \ge \min\left(\frac{|a_0\ t_1\ t_2|}{|\beta| + M_2}, R\right)$$

Combining (19) and (20), the desired result follows.

Proof of Theorem 2:- Consider

22. $$F(z) = (t_2 + z)(t_1 - z)P(z)$$

$$= (t_2 + z)(t_1 - z)(\ a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\ )$$

$$= -a_n z^{n+2} + (\ a_n\ (\ t_1 - t_2\ ) - a_{n-1}\ ) z^{n+1} + \cdots + a_0 t_1 t_2$$

Let

23. $$G(z) = z^{n+2} F(\tfrac{1}{z})$$

$$= -a_n + (a_n(t_1 - t_2) - a_{n-1})z +$$

$$(a_n t_1 t_2 - a_{n-1}(t_1 - t_2) a_{n-2}) z^2 + H(z)$$

Where

$$H(z) = z^2\{(a_n t_1 t_2 - a_{n-1}(t_1 - t_2) - a_{n-2}\ ) + \cdots + a_0 t_1 t_2 z^n\}$$

Clearly H(o)=0 = H(o), since $|H(z)| \le M_3 R^2$ for $|z| = R$

We first assume,

24 $|a_n| \leq MR^2 + R|a_n(t_1 - t_2) - a_{n-1}|$

Then by using Lemma 2. To H(z), it follows that

$$H|z| \leq \frac{M_3|z|}{R^2}\frac{M_3|z|}{M_3} Max_{|z|=R} |H(z)|$$

Hence from(23) we have

$$|G(z)| \geq |a_n| - |a_n(t_1 - t_2) - a_{n-1}||z| - \frac{|z|^2 M_3}{R^2} R^2$$

>0, if

$$M_3|z|^2 + |a_n(t_1 - t_2) - a_{n-1}||z| - |a_n| < 0$$

That is if,

25. $|z| < \frac{\{|a_n(t_1-t_2)-a_{n-1}|^2+4|a_n|M\}^{1/2} - |a_n(t_1-t_2)-a_{n-1}|}{2M_3}$

$= \frac{1}{r} \leq R$

If

$$|a_n(t_1 - t_2) - a_{n-1}|^2 + 4|a_n|M_3 \leq (2M_3 R + |a_n(t_1 - t_2) - a_{n-1}|^2$$

Which implies

$$|a_n| \leq M_3R^2 + R|a_n(t_1 - t_2) - a_{n-1}|$$

Which is true by (24)..Hence all the zeros of G(z) lie in $|Z| \geq r$. since

$$F(z) = z^{n+2}G(z),$$

It follows that all the zeros of F(z) and hence that of P(z) lie in $|z| \leq r$. We now assume.

26. $|a_n| \leq M_3R^2 + R\,|a_n(t_1 - t_2) - a_{n-1}|$,

then for $|z| \leq R$, from (23) it follows that

$$|G(z)| \geq |a_n| - |a_n(t_1 - t_2) - a_{n-1}|\,|z| - |H(z)|$$

$$\geq |a_n| - R\,|a_n(t_1 - t_2) - a_{n-1}| - R^2M_3$$

>0, by (26).

This shows that all the zeros of G(z) lie in $|Z| > R$, and hence all the zeros of $F(z) = z^{n+2}\, G(\frac{1}{z})$ lie in $|z| \leq \frac{1}{R}$, but all the zeros of P(z) are also the zeros of F(z) , therefore it follows that all the zeros of P(z) lie in

$$|Z| \leq \frac{1}{R}$$

27. From (25) amd (27) , we conclude that all the zeros of P(z) lie in

$$|Z| \leq \max(r, \frac{1}{R})$$

Which completes the proof of Theorem 2.

28. PROOF OF THEOREM 3. Consider the polynomial

F(z) = $(t_2 + z)(t_1 - z)P(z)$

$= (t_1 t_2 + (t_1 - t_2)z - z^2)(\, a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \,)$

$= -a_n z^{n+2} + (\, a_n\, (t_1 - t_2\,) - a_{n-1}\,) z^{n+1} + \cdots + (\, a_n t_1 t_2$

$+(\, a_{n-1}\, (t_1 - t_2\,) - a_{n-1}\,) z^{n+1} + \cdots +$

$(a_2 t_1 t_2 + a_1\, (t_1 - t_2\,) - a_0) z^2 + (a_1 t_1 t_2 + a_0\, (t_1 - t_2\,) z + a_0 t_1 t_2)$

We have,

G(z) = $z^{n+2} F(\frac{1}{z})$

$= r a_n + (a_n (t_1 - t_2) - a_{n-1}) z + (\, a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\,) z^2$

$+ \ldots + \big(a_1 t_1 t_2 + a_0\, (t_1 - t_2\,)\big) z^{n+1} + a_0 t_1 t_2\, z^{n+2}$

$= -a_n - \alpha z + (a_n (t_1 - t_2\,) + \alpha - a_{n-1}) z +$

$(a_n t_1 t_2 + a_{n-1}\, (t_1 - t_2\,) - a_{n-2}) z^2 + \ldots +$

$\big(a_1 t_1 t_2 + a_0\, (t_1 - t_2\,)\big) z^{n+1} + a_0 t_1 t_2\, z^{n+2}$

29. $= -a_n - \alpha z + H(z)$

Where

H(z) = $\{(a_n(\, t_1 - t_2\,) + \alpha - a_{n-1}\,) z + \cdots + a_0 t_1 t_2\, z^{n+2}\,)\}$

We first assume that

30. $|a_n| \geq |\alpha| R + M_4$

Then for $|z| < R$, we have

31. $|G(z)| \geq |a_n| - |\alpha||z| - |H(z)|$

Since

$|H(z)| \leq M_4\ for\ |z| \leq R$

Therefore for $|z|$<R, from (31) with the help of (30), we have

$|G(z)| > |a_n| - |\propto| R - M_4$

therefore , all the zeros of G(z) lie in $|z| \geq R$, in this case. Since F(z) = z^{n+2} $G\left(\frac{1}{z}\right)$ therefore all the zeros of G(z) lie in $|Z| \leq \frac{1}{R}$. As all the zeros of P(z) are the zeros of F(z), it follows that all the zeros of P(z) lie in

32. $|z| \leq \frac{1}{R}$

Now we assume $|a_n| < |\alpha| R + M_4$, clearly H(o) = 0 and H(o) = $(a_n(\, t_1 - t_2\,) - \alpha - a_{n-1}\,)$, Since by (11), $|H(z)| \leq M_4$, for $|z| = R$, therefore it follows by Lemma2. that

$$|H(z)| \leq \frac{M_4\ |z|}{R^2} \left(\frac{M_{4\ |z|} + R^2\ |a_n(\, t_1 - t_2) - a_{n-1} + \alpha|}{M_4 + |a_n(\, t_1 - t_2) - a_{n-1} + \alpha||z|} \right)$$

for $|z| \leq R$

Using this in (31), we get,

$$|G(z)| \geq |a_n| - |\alpha||z| - \frac{M_{4\ |z|}}{R^2} \left(\frac{M_{4\ |z|} + R^2\ |a_n((\, t_1 - t_2) - a_{n-1} + \alpha|}{M_4 + |a_n((\, t_1 - t_2) - a_{n-1} + \alpha||z|} \right)$$

>0, if

$$|a_n| R^2 (M_4 + |a_n((\, t_1 - t_2) - a_{n-1} + \alpha||z|) - |z|\{|\alpha| R^2 (M_4 + |a_n(\, t_1 - t_2) - a_{n-1} + \alpha||z|) + M_4(M_4|z| + R^2 |a_n(\, t_1 - t_2) - a_{n-1} + \alpha|\,)\,\} > 0,$$

this implies,

$$(|\alpha| R^2\, |a_n(\, t_1 - t_2\,) - a_{n-1} + \alpha| + M_4^2)|z|^2 + \{\, |\alpha| R^2 M_4 + R^2 M_4 |a_n(t_1 - t_2) - a_{n-1} + \propto| \ldots R^2 M_4 |a_n(\, t_1 - t_2\,) - a_{n-1} + \alpha|\}\, |z| - |a_n| R^2 M_4 < 0,$$

This gives,

$|G(z)| > 0, if$

$$|z| < [(|\alpha| R^2 M_4 + R^2(\, M_4 - |a_n|) |a_n(\, t_1 - t_2) - a_{n-1} + \alpha|\,)^2 + \ldots \ldots 4\big(|\alpha| R^2 |a_n(\, t_1 - t_2) - a_{n-1} + \alpha| + {M_4}^2\big)(|a_n| R^2 M_4)]$$

$$- \frac{(\,|\alpha| R^2 M_4 + R^2 (M_4 - |a_n|)\,) |a_n(\, t_1 - t_2) - a_{n-1} + \alpha|}{2\big(|\alpha| R^2 |a_n(\, t_1 - t_2) - a_{n-1} + \alpha| + {M_4}^2\big)}$$

$= \frac{1}{r_1}$,

So, $|z| < R, if$

$(|\alpha| R^2 M_4 + (\, M_4 - |a_n| R^2) |a_n(\, t_1 - t_2) - a_{n-1} + \alpha|\,)^2 +$

$$4\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+M_4{}^2\}\{|a_n|R^2M_4\}$$

$$<\{2|\alpha||a_n(t_1-t_2)-a_{n-1}+\alpha|R+|\alpha|R^2M_4+2RM_4{}^2$$

$$+(M_4-|a_n|R^2)|a_n(t_1-t_2)-a_{n-1}+\alpha|)^2\}$$

That is if,

$$|a_n|R^2M_4\leq R^2\left(|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+M_4{}^2\right)+$$

$$(|\alpha|R^2M_4)+(M_4-|a_n|R^2)|a_n(t_1-t_2)-a_{n-1}+\alpha|)R$$

Which gives

$$|a_n|M_4\leq M_4\{|\alpha|R+M_4-|a_n|+R\big||\alpha|R^2\big||a_n(t_1-t_2)-a_{n-1}+\alpha|+M_4{}^2|$$

$$\{|\alpha|R^2+M_4-a_n\}$$

Which is true because $a_n<|\alpha|R+M_4$,

Consequently , all the zeros of G(z) lie in $|z|\geq\frac{1}{r_1}$, as $F(z)=z^{n+1}G(\frac{1}{z})$, we conclude that all the zeros of F(z) lie $|z|\leq r_1$, since every zero of P(z) is also a zero of F(z), it follows that all the zeros of P(z) lie in

33. $$|z|\leq r_1$$

Combining this with (32) it follows that all the zeros of P(z) lie in

34. $$|z|\leq max\left(r_1,\frac{1}{R}\right)$$

Again from (28) it follows that

$$F(z)=|a_0t_1t_2-\beta z+(a_1t_1t_2(t_1-t_2)+\beta)z+\cdots+(a_n(t_1-t_2)$$

$$-.a_{n-1})z^{n+2}-a_nz^{n+2}|$$

35. $$\geq|a_0t_1t_2|-|\beta||z|-|T(z)|$$

Where,

$$T(z)=a_nz^{n+2}+(a_n(t_1-t_2)-.a_{n-1})z^{n+1}+..+(a_0t_1t_2+a_0(t_1-t_2)+\beta)z$$

Clearly T(o)=0, and T(o) $(a_1t_1t_2+a_0(t_1-t_2)+\beta)$
Since by (12) , $|T(z)|\leq M_5$ for $|z|=R$
using Lemma 2. To T(z), we have

$$F(z)\geq\left|a_0t_1t_2|-|\beta||z|-\frac{M_5|z|}{R^2}(M_5|z|+R^2|a_1t_1t_2|+a_0(t_1-t_2)+\beta)|z|\right|$$

$$=-\left(R^2|\beta||a_1t_1t_2|+a_0(t_1-t_2)+\beta|+M_5{}^2\right)|z|^2+$$

$$\{|a_1t_1t_2|+a_0(t_1-t_2)+\beta|(R^2|a_0t_1t_2|-R^2M_5)-|\beta|R^2M_5\}|z|+M_5R^2|a_1t_1t_2|$$

$$R^2(M_5+|a_1t_1t_2+a_0(t_1-t_2)+\beta||z|)$$

>0, if,

$$\left(R^2|\beta||a_1t_1t_2|+a_0(t_1-t_2)+\beta|+M_5{}^2\right)|z|^2-\{|a_1t_1t_2|+a_0(t_1-t_2)+\beta|$$

$$(R^2|a_0t_1t_2|-R^2M_5)-|\beta|R^2M_5\}|z|-R^2M_5|a_0|t_1t_2<0,$$

Thus, $|F(z)|>0, if$

$$|z|<-R^2\{|a_1t_1t_2+a_0(t_1-t_2)+\beta|(|a_0|t_1t_2-M_5)\big||\beta|M_5\}$$

$$-\{R^4(|a_1 t_1 t_2| + a_0 (t_{1-} t_2) + \beta |(|a_0| t_1 t_2 - M_5)||\beta| M_5)^2$$

$$\frac{+4(R^2|\beta||a_1 t_1 t_2 + a_0 (t_{1-} t_2) + \beta| + {M_5}^2)(R^2 M_5 |a_0| t_1 t_2)\}^{1/2}}{2(R^2|\beta||a_1 t_1 t_2| + a_0 (t_{1-} t_2) + \beta| + {M_5}^2)} \; r_z$$

Thus it follows by the same reasoning as in Theorem 1, that all the zeros of F(z) and hence that of P(z) lie in

36. $|z| \geq \min(r_{2,} R)$

Combining (34) and (36) , the desired result follows.

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