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Some New Families on H-Cordial Graphs

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Abstract: *In this paper some new families on H-cordial graphs are investigated. We prove that the graphs obtained by duplication of vertices as well as edges in cycle C_n, $n \geq 3$ admits H_2-Cordial labeling. In addition to this we derive that the joint sum of two copies of cycle C_n of even order, a triangle and quadruple in C_r and C_{r+1} respectively for each $r \geq 3$ and complete graph K_n, $n \equiv 1(mod4)$ are H_2-Cordial labeling. The shadow graph of path P_n for even n and spilt graph of C_n for even are H-cordial labeling.*

Keywords: *H-cordial labeling, H_2-cordial labeling, joint sum, shadow graphs.*

I. INTRODUCTION

We begin with simple, finite, connected and undirected graph G= (V (G), E (G)) with p vertices and q edges. For all other standard terminology and notations we follow (Harary, F.1972). We will provide brief summary of definitions and other information which serve as prerequisites for the present investigations.

1.1. Definition: Duplication of a vertex V_k of a graph G produces a new graph G_1 by adding a new vertex V_k' in such a way that $f(v_k) = f(v_k')$.

1.2. Definition : For a graph G the split graph is obtained by adding to each vertex v, a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted by spl (G).

1.3. Definition: Consider a cycle C_n. Let $e_k = V_k V_{k+1}$ be an edge in it, and let $e^1 = V_k V_k'$, $e'' = V_{k+1} V_{k+1}'$ be the new edges incident with V_k and V_{k+1}. The duplication of an edge e_k by an new edge, $e_k' = V_k' V_{k+1}'$ produces a new graph G in such a way that $f(V_k') = f(V_{k+1}')$.

1.4. Definition: Consider two copies of C_n, connect a vertex of the first copy to a vertex of second copy with a new edge, the new graph obtained is called the joint sum of C_n.

1.5. Definition: The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G′ and G″. Join each vertex V′ in G′ to the neighbours of the corresponding vertex V″ in G″.

1.6. Definition: If the vertices are assigned values subject to certain conditions then it is known as graph labeling.
The concept of an H-cordial graph is introduced by I-Cahit in 1996 (Bulletin of the ICA).
In [2] proved that if f is an assignment of integer numbers to the edges and vertices of a given graph G such that for each vertex v

$$f(v) = \sum_{e \epsilon I(v)} f(e) \text{ then } \sum_{V \epsilon V(G)} f(v) = 2 \sum_{e \epsilon E(G)} f(e).$$

1.7. Definition: Let G = (V (G), E (G)) be a graph. A mapping f: E (G) → {1, -1} is called H-cordial, if there exists a positive constant k, such that for each vertex v, |f (v)| = k, and the following two conditions are satisfied. $|e_f(1) - e_f(-1)| \leq 1$ and $|v_f(k) - v_f(-k)| \leq 1$, where v(i) and e(j) are respectively the number of vertices labeled with i and the number of edges labeled with j.
A graph G is called to be H-cordial, if it admits an H-cordial labeling.

1.8. Definition: An assignment f of integer labels to the edges of a graph G is called to be a H_k – cordial labeling, if for each edge e and each vertex v of G we have $1 \leq |f(e)| \leq k$ and $1 \leq |f(v)| \leq k$ and for each with $1 \leq i \leq k$ we have $|e_f(i) - e_f(-i)| \leq 1$ and $|v_f(i) - v_f(-i)| \leq 1$. A graph G is called to be H_k – cordial if it admits a H_k – cordial labeling.
Ghebleh and Khoeilar proved that K_n is H-cordial if and only if n≡0 or 3(mod 4) and n≠3. Wn is H-cordial if and only if n is odd.

K_n is H_2 – cordial if n≡0 or 3(mod 4) and K_n is not H_2-cordial if n≡1(mod 4).

C_n is not H-cordial and H_2 – cordial labeling. In [2] showed that triangle and quadruple C_r and C_{r+1} respectively for each $r \geq 3$ are not H_2 – cordial labeling. In this paper we duplicated the vertices and edges of those graphs and obtained a H_2-cordial labeling.

II. Main Results

2.1. Theorem: The graph obtained by duplication of two vertices in a graph of C_n, $n \geq 3$ admits a H_2 – cordial labeling.

Proof: Let v_1, v_2…………….. v_n be the vertices of the cycle C_n. Let G be the graph obtained by duplicating two vertices that lies between the edges of the labels +1 and -1.
Without loss of generality let the vertices be v_1 and v_2, the newly added vertices by v_1' v_2' and the edges be 'e' and e0.
E (G) = {E(Cn), e and e0 } where e = $v_1 v_1'$ and e0 = $v_2 v_2'$.
To define f : E(G) → {1, -1} two cases are to be considered.
Case (i) n is even.

$f(e_i)$ = 1, 1 ≤ i ≤ n/2

= -1, n/2+1 ≤ i ≤ n

f(e) = -1, $f(e_0)$ = 1.

Case(ii) n is odd

$f(e_i)$ = 1, 1 ≤ i ≤ n-1/2

= -1, $\frac{n-1}{2}$ + 1 ≤ i ≤ n

In view of the above defined labeling pattern f satisfies the condition for H_2- cordial labeling as shown in Figure 1. The vertex and the edge conditions are given in Table 1 and Table 2.

2. 2 Illustration: Figure 1 shows H_2-coridal labeling of the graph obtained by duplication of vertices in C_{12}

2.3 Theorem: The graph obtained by duplication of edges in C_n, n≥3 admits a H_2-cordial labeling.

Proof: Let v_1, v_2, ……………… v_n be the vertices of the cycle C_n.
Let G be the graph obtained by duplicating edges of C_n that lies between the labels of the vertices 0 and |2|.
Without loss of generality, let the edge be $e_1=v_1v_2$ and the new edges incident with v_1 and v_2 are $e_0' = v_1v_1', e_0'' = v_2v_2'$. The duplication of an edge e_1 by another new edge $e_1' = v_1'v_2'$. Similarly define the other edge that lies between the vertices of the labels 0 and |2|.

Define f : E(G) → {1, -1}; two cases are to be considered.

Case (i) when n is even

$f(e_i)$ = 1, 1 ≤ i ≤ n/2

= -1, $^n/_2$+1 ≤ i ≤ n

$f(e_1)$ = -1, $f(e_1')$ = 1, $f(e_0')$ = 1, $f(e_0'')$ = 1

Similarly f (e_2) = 1, $f(e_2')$ = -1, f(e′) = -1, f(e′′) = -1

Case (ii) when n is odd

f (e_i) = 1, 1 ≤ i ≤ n-1/2

= -1, $^n/_2$+1 ≤ i ≤ n

$f(e_1)$ = 1, $f(e_1')$ = -1, $f(e_0')$ = -1, $f(e_0'')$ = -1

Similarly $f(e_6)$ = -1, $f(e_6')$ = 1, f(e′) = 1, f(e′′) = 1

In view of the above defined labeling pattern f satisfies the condition for H_2-cordial labeling as shown in Figure 2. The edge and the vertex conditions are given in Table 3 and Table 4.

2.4 Illustration: H_2-cordial labeling of the graph obtained by duplication of the edge in C_{11} is shown in figure 2.

2.5 Theorem: Joint sum of two copies of C_n, n≥3 of even order produce an H_2-cordial labeling graph.

Proof: Let the vertices of the first copy of C_n by v_1, v_2, ……….. v_n and second copy by v_1', v_2'……….. v_n', e_i and e_i' where i ≤ i ≤ n be the corresponding edges.
Join the copies of Cn with new edges and let G be the resultant graph. Without loss of generality we assume that the new edges be e = v_1v_1' and e′ = v_2v_2'.

Define $f : E(G) \rightarrow \{1,-1\}$.
When n is even

$f(e_i) = 1, \quad 1 \leq i \leq n/2$

$= -1, \quad n/2+1 \leq i \leq n$; $\quad f(e) = -1, f(e') = 1$

In view of the above defined labeling pattern f satisfies the condition for H_2-cordial labeling as shown in Figure 3. The edge and the vertex conditions are given in Table 5.

2.6 Illustration: H_2-cordial labeling of joint sum of two copies of C_{10} is shown in Figure 3.

2.7 Theorem: The split graph of C_n, $n \geq 3$ of even order is H-cordial.

Proof: Let v_1, v_2 ……………. V_n be the vertices of cycle C_n and v_1', v_2', ……………… v_n' be the newly added vertices when n is even.
Let G be the split graph of cycle C_n with $V(G) = \{v_i, v_i', 1 \leq i \leq n\}$ and
$E(G) = \{v_i, v_{i+1}, 1 \leq i \leq n-1, v_nv_1, v_i'v_{i+1},$

$1 \leq i \leq n-1, v_n'v_1, v_iv_{i+1}' \; 1 \leq i \leq n-1, v_nv_1'\}$

To define $f : E(G) \rightarrow \{1, -1\}$

$f(e_i) = f(v_i, v_{i+1}) = 1, 1 \leq i \leq n/2$

$= -1, n/2+1 \leq i \leq n$

$f(v_nv_1) = -1$

<u>For 1 ≤ i ≤ n/2</u>

$f(v_i, v_{i+1}') = 1$ if $i \equiv 0, 1 \pmod 4$

$= -1$ if $i \equiv 2, 3 \pmod 4$

<u>For n/2 +1 ≤ i ≤ n</u>

$f(v_i, v_{i+1}') = -1$ if $i \equiv 1, 2 \pmod 4$

$= 1$ if $i \equiv 0, 3 \pmod 4$

$f(v_n, v_1') = 1$

<u>For 1 ≤ i ≤ n/2</u>

$f(v_i' v_{i+1}) = 1$, if $i \equiv 1, 2 \pmod 4$

$= -1$, if $i \equiv 0,3 \pmod 4$

<u>For n/2 +1 ≤ i ≤ n</u>

$f(v_i' v_{i+1}) = 1$ if $i \equiv 0, 1 \pmod 4$

$= -1$ if $i \equiv 2, 3 \pmod 4$

$f(v_n', v_1) = 1$

In view of the above defined labeling pattern f satisfies the condition for H-cordial labeling as shown in Figure 4. The edge and the vertex conditions are given in Table 6.

2.8 Illustration: H-cordial labeling of split graph of cycle C_8 is shown in figure 4.

2.9 Theorem: $D_2(P_n)$ is H-cordial labeling for even n.

Proof **:** Let P_n', P_n'' be two copies of path P_n. We denote the vertices of first copy of P_n by v_1', v_2', ………. v_n' and the second copy by v_1'', v_2'' ……………. v_n''. Let G be $D_2(P_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n-4$.

Define f: E(G) → {1, -1}

n ≡ 0, 2 (mod 4)

$f(v_i', v_{i+1}')$ = 1, $1 \le i \le n/2$

= -1, $n/2 + 1 \le i \le n-1$

$f(v_i'', v_{i+1}')$ = 1, $1 \le i \le n-2$

= -1, $n-2 \le i \le n$

<u>For 1 ≤ i ≤ n-1</u>

$f(V_i' V_i+_1'')$ = 1, i ≡ 0, 1 (mod 4)

= -1, i ≡ 2, 3 (mod 4)

$f(v_{n-1}' v_n'')$ = -1

<u>For i ≤ i ≤ n-1</u>

$f(v_i'', v_{i+1}'')$ = 1, i ≠ j, j = 2,3,…..n-1

= -1, i = j

In view of the above defined labeling pattern f satisfied the condition for H-cordial labeling for even n, as shown in Figure 5. The edge and the vertex conditions are given in Table 7.

2.10 Illustration: H-cordial labeling of D_2 (P_6) is shown in figure 5.

The duplication of Triangle and quadruple, C_r and C_{r+1}, r ≥ 3 admits a H_2-cordial labeling.

Proof: Consider a triangle C_3 and quadruple C_4. Let the vertices be v_1, v_2………… v_7 and the corresponding edges are e_1, e_2 ………. e_{13}.

Take an edge $e_1 = v_1v_2$ be the duplicating edge that lies between the labels of the vertices 0 and |2|.

Define f : E(G) → {1, -1}

E(G) = {E(C_3 & C_4) e′, e′′} where $e' = v_1v_1'$

$e'' = v_2v_2'$, $e_1' = v_1' v_2'$

The newly added edges be e_1', e′ and e′′.

$f(v_i, v_{i+1}) = 1$, $1 \le i \le n-2$

$f(v_1, v_n) = -1$ $f(v_2, v_{n-1}) = -1$

$f(v_3, v_{n-1}) = -1$ $f(v_6, v_n) = -1$

$f(e') = f(v_1v_1') = -1$

$f(e'') = f(v_2v_2') = -1$

$f(e_1^1) = f(v_1^1 v_2^1) = -1$

In view of the above defined labeling pattern f satisfies the condition for H_2-cordial labeling as shown in Figure 6. The edge and the vertex conditions are given in Table 8.

2.11 Illustration: H_2-cordial labeling of the graph obtained by duplication of the edge in C_3 & C_4 is shown in figure 6.

2.11 Theorem: The graph obtained by duplication of an edge of the complete graph k_n, n ≡ 1 (mod 4) admits a H_2-cordial labeling.

Proof: Let v_1, v_2, ……………… v_n be the vertices of the complete graph K_n. Let G be the graph obtained by duplicating an edge that lies between the vertices of the labels 0 and |2|.
Without loss of generality let this edge be e_1 and the newly added edge be e_1'.
$E(G) = \{E(K_n), e_1', e' \text{ and } e''\}$ where
$e_1' = v_1' v_2'$, $e' = v_1 v_1'$, $e'' = v_2 v_2'$
By the definition of duplication of an edge we have $f(v_k') = f(v_{k+1}')$.
Define $f : E(G) \rightarrow \{1, -1\}$

For $1 \leq i \leq n-1$

$f(v_i, v_{i+1}) = 1, \quad i \equiv 1, 3 \pmod 4$

$= -1, \quad i \equiv 0, 2 \pmod 4$

$f(e_1') = -1$

$f(e') = -1$

$f(e'') = -1$

$f(v_1, v_{n-2}) = -1$

$f(v_1, v_{n-1}) = 1, \quad f(v_2, v_n) = 1$

$f(v_2, v_{n-1}) = 1$

$f(v_3, v_n) = 1$

:

:

:

In view of the above defined labeling pattern f satisfies the condition for H_2-cordial labeling as shown in Figure 7. The edge and the vertex conditions are given in Table 9.

2.13 Illustration: H_2-cordial labeling of the graph obtained by duplicating of an edge K_5 is shown figure 7.

TABLE 1

n	Vertex Condition	Edge Condition
even	$v_f(1) = v_f(-1) = n-(n-2)$ $v_f(2) = v_f(-2) = n-2/2$	$e_f(1)=e_f(-1)=\frac{n+2}{2}$ $e_f(2)=e_f(-2)= 0$

TABLE 2

n	Vertex Condition	Edge Condition
odd	$v_f(1) = v_f(-1) = n-(n-2)$ $v_f(-2) = v_f(2)+1 = n-1/2$	$e_f(-1)=e_f(1)+1=\frac{n+3}{2}$ $e_f(2)=e_f(-2)= 0$

TABLE 3

n	Vertex Condition	Edge Condition
Odd	$v_f(1) = v_f(-1) = n-(n-2)$ $v_f(-2) = v_f(+2)+1 = n+1/2$	$e_f(-1)=e_f(1)+1 = n+7/2$ $e_f(-2)=e_f(2)=0$

TABLE 4

n	Vertex Condition	Edge Condition
Even	$v_f(1) = v_f(-1) = n-(n-2)$ $v_f(-2) = v_f(-2)+1 = n/2$	$e_f(1)=e_f(-1) = n+6/2$ $e_f(2)=e_f(-2)=0$

TABLE 5

n	Vertex Condition	Edge Condition
Even	$v_f(1) = v_f(-1) = n-(n-2)$ $v_f(2) = v_f(-2) = n+6/2$	$e_f(1)=e_f(-1) = n+12/2$ $e_f(2)=e_f(-2)=0$

TABLE 6

n	Vertex Condition	Edge Condition
even	$v_f(2) = v_f(-2) = n$	$e_f(1)=e_f(-1) = \frac{3n}{2}$

TABLE 7

n	Vertex Condition	Edge Condition
Even	$v_f(-2) = v_f(2) +1 = \frac{n+6}{2}$	$e_f(1)=e_f(-1) = 2n-2$

TABLE 8

n	Vertex Condition	Edge Condition
Odd	$v_f(1) = v_f(-1) = n-1/2$ $v_f(-2) = v_f(2)+1 = 1$	$e_f(-1) = e_f(1)+1= \frac{n+9}{2}$ $e_f(2) = e_f(-2) = 0$

TABLE 9

n	Vertex Condition	Edge Condition
n≡1(mod4)	$v_f(1) = v_f(-1) = n-(n-1)$ $v_f(2) = v_f(-2)+1 = n+1/2$	$e_f(1) = e_f(-1)+1=n+9/2$ $e_f(2) = e_f(-2) = 0$

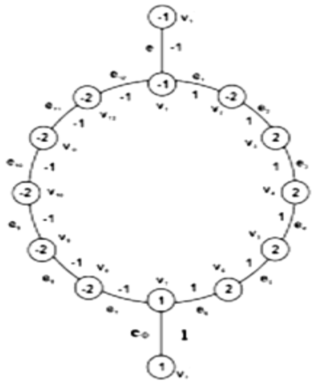


Figure 1 H_2-coridal labeling of the graph obtained by duplication of the vertices in C_{12}

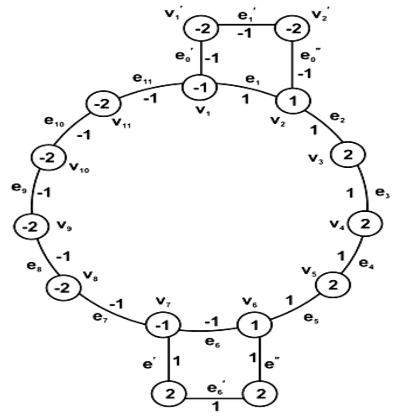


Figure 2: H_2-cordial labeling of the graph
by duplication of the edge in C_{11}.

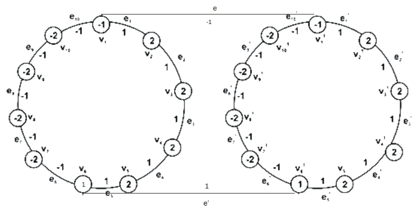


Figure 3 : H_2-cordial labeling of joint sum of two copies of C_{10}.

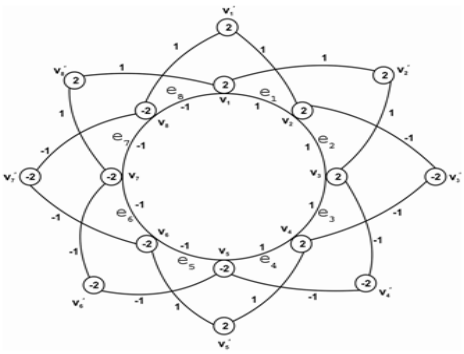


Figure 4 : H-cordial labeling of split
graph of cycle C_8

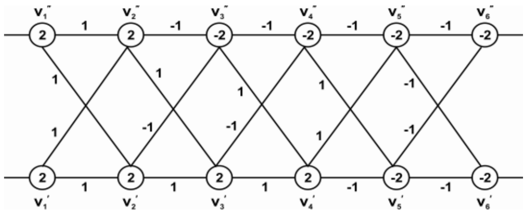


Figure 5: H-cordial labeling of D_2 (P_6)

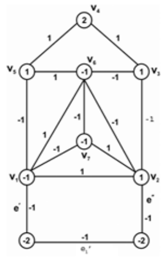


Figure 6: H_2-cordial labeling of the graph obtained by duplication of the edge in C_3 & C_4

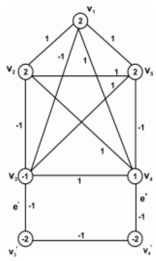
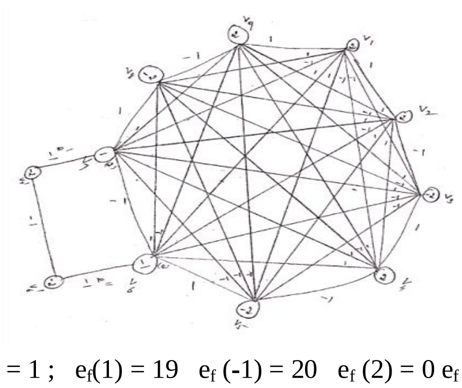


Figure 7 : H_2-cordial labeling of the graph obtained by duplication of an edge K_5.

Example : 1

Show that K_9 is a H_2- cordial labeling by using theorem 2.2



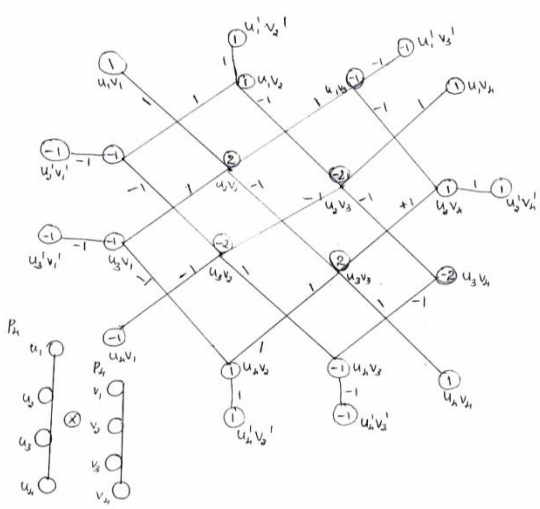
$V_f(2) = 4$, $V_f(-2) = 5$ $V_f(1) = 1$ $V_f(-1) = 1$; $e_f(1) = 19$ $e_f(-1) = 20$ $e_f(2) = 0$ $e_f(-2) = 0$

For each i $1 \le i \le 2$; $|V_f(1) - V_f(-1)| +$
$|V_f(2) - V_f(-2)| = |1-1| + |4-5| \le 1$

For each i, $1 \le i \le 2$; $| e_f(1) - e_f(-1)| +$
$| e_f(2) - e_f(-2)| = 19-20| + |0-1| \le 1$

$\therefore$ K_9 is H_2 Cordial labeling.

Example 2: Show that $P_4 \otimes P_4$ is a H_2 Cordial labeling by using theorem 2.1



$V_f(2) = 2$, $V_f(-2) = 3$ $V_f(1) = 9$ $V_f(-1) = 9$; $e_f(1) = 13$ $e_f(-1) = 13$ $e_f(2) = 0$ $e_f(-2) = 0$

For each i $1 \le i \le 2$; $|V_f(1) - V_f(-1)| +$
$|V_f(2) - V_f(-2)| = |9-9| + |2-3| \le 1$

For each i $1 \le i \le 2$;$| e_f(1) - e_f(-1)| +$
$| e_f(2) - e_f(-2)| = |13-13| + |0-0| \le 1$

$\therefore$ $P_4 \otimes P_4$ is a H_2 Cordial labeling.

III. Concluding Remarks

Here we investigate seven new families of H-cordial graphs generated by different graph operations. To investigate similar results for other graph families and in the context of different graph labeling problems is an open area of research.

IV. ACKNOWLEDGMENT

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