Common fixed point theorems in intuitionistic fuzzy metric space

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Abstract: *In this paper, we give the Mizoguchi-Takahashi's theorems into intuitionistic fuzzy metric space.*

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I. Introduction

The notion of fuzzy sets was introduced by A. Zadeh [11] in 1965. George and Veeramani [5] introduced the fuzzy metric space. Mihet [7] obtained some new results of modifying the notion of convergences in fuzzy metric space. Park [9] used the idea of intuitionistic fuzzy sets with the help of t-norm and t-conorm as a generalization of fuzzy metric space and introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and also proved the Baire's theorem.

Alaca et al. [1] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space similar to Park [9] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [6]. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well-known theorems of Banach [2] and Edelstein [3]. Many authors have studied the concept of intuitionistic fuzzy metric space and its applications [4, 10, 12, 13, 14].

The purpose of this paper, is to prove the Mizoguchi-Takahashi's [6] theorems in intuitionistic fuzzy metric space.

Definition 1.1: A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if $*$ satisfy the following conditions:
(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0, 1]$;
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.2: A binary operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\Diamond$ satisfy the following conditions:
(1) $\Diamond$ is commutative and associative;
(2) $\Diamond$ is continuous;
(3) $a \Diamond 0 = a$ for all $a \in [0, 1]$;
(4) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.3: (Alaca et al.) [1] A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:
(1) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and $t > 0$;
(2) $M(x, y, 0) = 0$ for all $x, y \in X$;
(3) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(4) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(5) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(6) for all $x, y \in X$ and $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;
(7) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(8) $N(x, y, 0) = 1$ for all $x, y \in X$
(9) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(10) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(11) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(12) for all $x, y \in X$ and $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;
(13) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respective to t, respectively.

Definition 1.4: Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space, then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$ $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$

(b) a sequence $\{x_n\}$ in X is said to be convergent sequence if, for all $t > 0$ and $p > 0$ $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$

(c) An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

(d) An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be compact if every sequence in X contains a convergent subsequence.

Example 1.1: Let (X, d) be a metric space. Define t-norm $a * b = \min\{a, b\}$ and t-co-norm $a \lozenge b = \max\{a, b\}$ and for all $a, b \in X$ and $t > 0$.
Let us define $M(x, y, t) = t/(t + d(x, y))$ and $N(x, y, t) = d(x, y)/(t + d(x, y))$.
Then $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space.

II. Main Results

In 1989, Mizoguchi and Takahashi [8] proved the following fixed point theorem.

Theorem2.1: **(Mizoguchi and Takahashi)** Let (X, d) be a complete metric space and T a map from X into CB(X), where CB(X) is the class of all nonempty closed bounded subsets of X. Assume that

$$H(Tx, Ty) \le \alpha(d(x, y))\, d(x, y)$$

For all $x, y \in X$, where α is a function from $[0,\infty)$ into $[0, 1)$ satisfying $\limsup_{s\to t+0} \alpha(s) < 1$ for all $t \in [0,\infty)$. Then there exists $z \in X$ such that $z \in Tz$.

In fact, Mizoguchi-Takahashi's fixed point theorem is a generalization of Nadler's fixed point theorem [13] which extended the Banach contraction principle to multivalued maps, but its primitive proof is different.

Then above theorem can be proved in an intuitionistic fuzzy metric space as follows.

Theorem 2.2: Let $(X, M, N, *, \lozenge)$ be a complete intuitionistic fuzzy metric space with continuous t-norm * and continuous t-conorm $\lozenge$ defined by $t * t \ge t$ and
$(1 - t) \lozenge (1 - t) \le (1 - t)$ for all $t \in [0, 1]$ and $T: X \to$ CB(X) is multivalued map and
$\varphi: [0, \infty) \to [0, 1)$ is continuous function. There exists $0 < k < 1$ such that for all $x, y \in$ X
$M(Tx, Ty, kt) \ge M(x, y, t) * M(x, y, t)$ and $N(Tx, Ty, kt) \le N(x, y, t) \lozenge N(x, y, t)$, then T has fixed point in X.

Proof: Let $\{X_n\}$ be a sequence in X and $X_{m-1} = X_m$ for some m, T has a fixed point X_m. Suppose that $X_{n-1} \ne X_n$ then $M(x_n, \mathbf{x_{n+1}}, kt) = M(Tx_{n+1}, Tx_{n+2}, kt)$
$\ge M(x_{n+1}, x_{n+2}, t/k) * M(x_{n+1}, x_{n+2}, t/k) \ge M(x_{n+1}, x_{n+2}, t/k^2)$
and $N(x_{n+1}, x_{n+2}, kt) \le N(x_{n+1}, x_{n+2}, t/k^2)$
Hence for any positive integer p
$M(x_n, x_{n+p}, kt) \ge M(x_{n+1}, x_{n+2}, t/k) * \ldots$ p-times $\ldots * M(x_{p+1-n}, x_{p+2-n}, t/k^n)$
$N(x_n, x_{n+p}, kt) \le N(x_{n+1}, x_{n+2}, t/k) \lozenge \ldots$ p-times$\ldots \lozenge N(x_{p+1-n}, x_{p+2-n}, t/k^n)$
When $n \to \infty$ then $\lim_{n\to\infty} M(x_n, x_{n+1}, kt) \ge 1 * 1 * \ldots * 1 = 1$
and $\lim_{n\to\infty} N(x_n, x_{n+1}, kt) \le 0 \lozenge 0 \lozenge \ldots \lozenge 0 = 0$
It shows that $\{x_n\}$ is Cauchy sequence in X and so, by the completeness of X, $\{x_n\}$ converges to a point x, then $M(x_n, x, kt) \ge M(x_n, x, t/k^2)$
and $N(x_n, x, kt) \le N(x_n, x, t/k^2)$.
Let y be another fixed point in X and $x \ne y$ then
$M(x_n, y, kt) = M(Tx_n, Ty, kt) \ge M(x_n, y, t/k^2)$
and $N(x_n, y, kt) = N(Tx_n, Ty, kt) \le N(x_n, y, t/k^2)$ when $n \to \infty$ gives that $M(x_n, y, t/k^2) = 1$ and $N(x_n, y, t/k^2) = 0$ for all $t > 0$, therefore it shows that $x = y$ so x is the fixed point of T.

Theorem 2.3: Let $(X, M, N, *, \lozenge)$ be a compact intuitionistic fuzzy metric space with continuous t-norm * and continuous t-conorm $\lozenge$ defined by $t * t \ge t$ and
$(1 - t) \lozenge (1 - t) \le (1 - t)$ for all $t \in [0, 1]$ and $T: X \to$ CB(X) is multivalued map and
$\varphi: [0, \infty) \to [0, 1)$ is continuous function. There exists $0 < k < 1$ such that for all $x, y \in X$
$M(Tx, Ty, kt) \ge M(x, y, t) * M(x, y, t)$ and $N(Tx, Ty, kt) \le N(x, y, t) \lozenge N(x, y, t)$, then T has fixed point in X.

Proof: Let $\{X_n\}$ be a sequence in X and $X_{m-1} = X_m$ for some m, T has a fixed point X_m. Suppose that $X_{n-1} \ne X_n$ then $M(x_n, x_{n+1}, kt) = M(Tx_{n+1}, Tx_{n+2}, kt)$
$\ge M(x_{n+1}, x_{n+2}, t/k) * M(x_{n+1}, x_{n+2}, t/k) \ge M(x_{n+1}, x_{n+2}, t/k^2)$
and $N(x_{n+1}, x_{n+2}, kt) \le N(x_{n+1}, x_{n+2}, t/k^2)$
Hence for any positive integer p
$M(x_n, x_{n+p}, kt) \ge M(x_{n+1}, x_{n+2}, t/k) * \ldots$ p-times $\ldots * M(x_{p+1-n}, x_{p+2-n}, t/k^n)$

$N(x_n, x_{n+p}, kt) \leq N(x_{n+1}, x_{n+2}, t/k) \lozenge \ldots$ p-times $\ldots \lozenge N(x_{p+1-n}, x_{p+2-n}, t/k^n)$

When $n \to \infty$ then $\lim_{n\to\infty} M(x_n, x_{n+1}, kt) \geq 1 * 1 * \ldots * 1 = 1$

And $\lim_{n\to\infty} N(x_n, x_{n+1}, kt) \leq 0 \lozenge 0 \lozenge \ldots \lozenge 0 = 0$.

It shows that $\{x_n\}$ is Cauchy sequence in X, since X is compact so, $\{x_n\}$ has a convergent subsequence $\{x_{ni}\}$. Let $\lim_{i\to\infty} \{x_{ni}\} = y$. Now we assume that y, $Ty \notin \{x_n\}$. Since T is continuous for all x, y in X, then $\lim_{i\to\infty}(Tx_{ni}, Ty, t) \geq \lim_{i\to\infty} M(x_{ni}, y, t) = 1$ for each

$t > 0$, hence $\lim_{i\to\infty} Tx_{ni} = Ty$ similarly $\lim_{i\to\infty} T^2x_{ni} = T^2y$

(Now again assume that $Ty \neq Tx_{ni}$ for all i). Now we observe that

$M(x_{n1}, Tx_{n1}, t) < M(Tx_{n1}, T^2x_{n1}, t) <\ldots< M(Tx_{ni}, T^2x_{ni}, t) <\ldots< M(Tx_{ni+1}, T^2x_{ni+1}, t) <\ldots< 1$ for all $t > 0$. Thus $\{M(x_{ni}, Tx_{ni}, t)\}$ and $\{M(Tx_{ni}, T^2x_{ni}, t)\}$ are convergent sequences to a common limit, i.e. $M(y, Ty, t)$. It shows that $\{M(x_{ni}, Tx_{ni}, t)\}=\{M(Tx_{ni}, T^2x_{ni}, t)\}$ is contradiction. Hence $y = Ty$, is a common fixed point.

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