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A Special Type Of Differential Polynomial And Its Comparative Growth Properties

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Abstract: *Some new results depending upon the comparative growth rates of composite entire and meromorphic function and a special type of differential polynomial as considered by Bhooshnurmath and Prasad[3] and generated by one of the factors of the composition are obtained in this paper.*
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I. INTRODUCTION, DEFINITIONS AND NOTATIONS.

Let $\mathbb{C}$ be the set of all finite complex numbers. Also let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following two notations:

$$(i)\ \log^{[k]}x = \log(\log^{[k-1]}x) \quad for\ k = 1,2,3,\dots;\ \log^{[0]}x = x$$

and

$$(ii)\ \exp^{[k]}x = \exp\left(exp^{[k-1]}x\right)\ for\ k = 1,2,3,\dots;\ \exp^{[0]}x = x.$$

The following definitions are frequently used in this paper:

Definition 1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2 *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f are defined as follows*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 3 *The type σ_f of a meromorphic function f is defined as follows*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Definition 4 *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f of finite lower order λ_f if*

(i) $\lambda_f(r)$ *is non-negative and continuous for* $r \geq r_0$, *say*

(ii) $\lambda_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\lambda_f'(r+0)$ *and* $\lambda_f'(r-0)$ *exists,*

(iii)

$$\lim_{r\to\infty} \lambda_f(r) = \lambda_f,$$

(iv)

$$\lim_{r\to\infty} r\lambda_f'(r)\log r = 0$$

and

(v)

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1.$$

Definition 5 *Let 'a' be a complex number, finite or infinite. The Nevanlinna's deficiency and Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, \ldots, n_{kj}$ $(k \geq 1)$ be non-negative integers such that for each j, $\sum_{i=0}^{k} n_{ij} \geq 1$. We call

$$M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$$

where $T(r, A_j) = S(r,f)$, to be a differential monomial generated by f. The numbers

$$\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$$

and

$$\Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij}$$

are called the degree and weight of $M_j[f]$ {cf. [4]} respectively. The expression

$$P[f] = \sum_{j=1}^{s} M_j[f]$$

is called a differential polynomial generated by f. The numbers

$$\gamma_P = \max_{1\leq j\leq s} \gamma_{M_j}$$

and

$$\Gamma_P = \max_{1\leq j\leq s} \Gamma_{M_j}$$

are respectively called the degree and weight of $P[f]$ {cf. [4]}. Also we call the numbers

$$\underline{\gamma_P} = \min_{1\leq j\leq s} \gamma_{M_j}$$

and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_P} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial.

Bhooshnurmath and Prasad [3] considered a special type of differential polynomial of the form $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f and n = 0, 1, 2,…. In this paper we intend to prove some improved results depending upon the comparative growth properties of the composition of entire and meromorphic functions and a special type of differential polynomial as mentioned above and generated by one of the factors of the composition. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [9] and [5].

II. LEMMAS.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[1] If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r, f_o g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 *[2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f_o g) \geq T(\exp(r^\mu), f).$$

Lemma 3 *[3] Let $F = f^n Q[f]$ where Q[f] is a differential polynomial in f. If $n \geq 1$ then $\rho_F = \rho_f$ and $\lambda_F = \lambda_f$.*

Lemma 4 *Let $F = f^n Q[f]$ where Q[f] is a differential polynomial in f. If $n \geq 1$ then*

$$\lim_{r\to\infty} \frac{T(r, F)}{T(r,f)} = 1.$$

The proof of Lemma 4 directly follows from Lemma 3.

In the line of Lemma 3 we may prove the following lemma:

Lemma 5 *Let $F = f^n Q[f]$ where Q[f] is a differential polynomial in f. If n ≥ 1 then* $\overline{\rho}_F = \overline{\rho}_f$ *and* $\overline{\lambda}_F = \overline{\lambda}_f$.

Lemma 6 *For a meromorphic function f of finite lower order, lower proximate order exists.*

The lemma can be proved in the line of Theorem 1 [7] and so the proof is omitted.

Lemma 7 *Let f be a meromorphic function of finite lower order λ_f. Then for $\delta(> 0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of r.*

Proof. Since

$$\frac{d}{dr} r^{\lambda_f+\delta-\lambda_f(r)} = \{\lambda_f + \delta - \lambda_f(r) - r\lambda_f' \log r\} r^{\lambda_f+\delta-\lambda_f(r)-1} > 0$$

for sufficiently large values of r, the lemma follows.

Lemma 8 *[6] Let f be an entire function of finite lower order. If there exists entire functions a_i (i= 1, 2, 3,…, n; n ≤ ∞) satisfying $T(r, a_i) = o\{T(r, f)\}$ and*

$$\sum_{i=1}^{n} \delta(a_i, f) = 1,$$

then

$$\lim_{r\to\infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}.$$

III. THEOREMS.

In this section we present the main results of the paper.

Theorem 1 *Let f be a meromorphic function and g be an entire function satisfying*
(i) λ_f, λ_g are both finite and
(ii) for $n \geq 1, G = g^n Q[g]$. Then

$$\liminf_{r\to\infty} \frac{\log T(r, f_o g)}{T(r, G)} \leq 3. \rho_f . 2^{\lambda_g}.$$

Proof. If $\rho_f = \infty$, the result is obvious. So we suppose that $\rho_f < \infty$. Since $T(r, g) \leq \log^+ M(r, g)$, in view of Lemma 1 we get for all sufficiently large values of r that

$$T(r, f_o g) \leq \{1 + o(1)\} T(M(r, g), f)$$

i.e.,

$$\log T(r, f_o g) \leq \log\{1 + o(1)\} + \log T(M(r, g), f)$$

i.e.,

$$\log T(r, f_o g) \leq o(1) + (\rho_f + \varepsilon) \log M(r, g)$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log T(r, f_o g)}{T(r, g)} \leq (\rho_f + \varepsilon) \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, f_o g)}{T(r, g)} \leq \rho_f . \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \quad (1)$$

As by condition (v) of Definition 4

$$\liminf_{r\to\infty} \frac{T(r, g)}{r^{\lambda_g(r)}} = 1,$$

so for given $\varepsilon(0 < \varepsilon < 1)$ we get for a sequence of values of r tending to infinity that

$$T(r, g) \leq (1 + \varepsilon) r^{\lambda_g(r)} \quad (2)$$

and for all sufficiently large values of r,

$$T(r, g) > (1 - \varepsilon) r^{\lambda_g(r)} \quad (3)$$

Since

$$\log M(r, g) \leq 3T(2r, g)$$

{cf. [5]}, we have by (2), for a sequence of values of r tending to infinity,

$$\log M(r, g) \leq 3T(2r, g) \leq 3(1 + \varepsilon)(2r)^{\lambda_g(2r)}. \quad (4)$$

Combining (3) and (4), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} . \frac{(2r)^{\lambda_g(2r)}}{r^{\lambda_g(r)}}.$$

Now for any $\delta(> 0)$, for a sequence of values of r tending to infinity we obtain that

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} . \frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}} . \frac{1}{r^{\lambda_g(r)}}$$

i.e.,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.2^{\lambda_g+\delta} \qquad (5)$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r by Lemma 7. Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary, it follows from (5) that

$$\liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} \leq 3.2^{\lambda_g}. \qquad (6)$$

Thus from (1) and (6) we obtain that

$$\liminf_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,g)} \leq 3.\rho_f.2^{\lambda_g}. \qquad (7)$$

Now in view of (7) and Lemma 3, we get that

$$\liminf_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,G)} = \liminf_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,G)}$$
$$\leq 3.\rho_f.2^{\lambda_g}.$$

This proves the theorem.

Theorem 2 *Let f be meromorphic and g be entire such that $\rho_f < \infty, \lambda_g < \infty$ and for $n \geq 1, G = g^n Q[g]$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,f_o g)}{\log T(r,G)} \leq 1.$$

Proof. Since

$$T(r,g) \leq \log^+ M(r,g),$$

in view of Lemma 1, we get for all sufficiently large values of r that

$$\log T(r,f_o g) \leq \log T(M(r,g),f) + \log\{1+o(1)\}$$

i.e.,

$$\log T(r,f_o g) \leq \left(\rho_f+\varepsilon\right)\log M(r,g) + o(1)$$

i.e.,

$$\log^{[2]} T(r,f_o g) \leq \log^{[2]} M(r,g) + O(1). \qquad (8)$$

It is well known that for any entire function g,

$$\log M(r,g) \leq 3T(2r,g) \qquad \{cf.[5]\}.$$

Then for $0<\varepsilon<1$ and $\delta(>0)$, for a sequence of values of r tending to infinity it follows from (5) that

$$\log^{[2]} M(r,g) \leq \log T(r,g) + O(1). \qquad (9)$$

Now combining (8) and (9), we obtain for a sequence of values of r tending to infinity that

$$\log^{[2]} T(r,f_o g) \leq \log T(r,g) + O(1)$$

i.e.,

$$\frac{\log^{[2]} T(r,f_o g)}{\log T(r,g)} \leq 1. \qquad (10)$$

As by Lemma 4,

$$\lim_{r\to\infty}\frac{\log T(r,g)}{\log T(r,G)}$$

exists and is equal to 1, then from (10) we get that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,f_o g)}{\log T(r,G)} = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f_o g)}{\log T(r,g)}.\lim_{r\to\infty}\frac{\log T(r,g)}{\log T(r,G)}$$
$$\leq 1.1 = 1.$$

Thus the theorem is established.

Remark 1 The *condition $\rho_f < \infty$ is essential in Theorem 2 which is evident from the following example.*

Example 1 *Let $f = \exp^{[2]} z$ and $g = \exp z$. Then $f_o g = \exp^{[3]} z$ and for $n \geq 1, G = g^n Q[g]$. Taking $n = 1, A_j = 1, n_{0j} = 1$ and $n_{1j} = \cdots = n_{kj} = 0$; we see that $G = \exp(2z)$. Now we have*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]}(\exp^{[2]} r)}{\log r} = \infty$$

and

$$\lambda_g = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[2]}(\exp r)}{\log r} = 1.$$

Again from the inequality

$$T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f)$$

{cf. p.18, [5]} we obtain that

$$T(r,G) \leq \log M(r,G) = \log(\exp 2r)$$

i.e.,

$$\log T(r,G) \leq \log r + O(1)$$

and

$$T(r,f_o g) \geq \frac{1}{3}\log M\left(\frac{r}{2},f_o g\right) = \frac{1}{3}\exp^{[2]}\left(\frac{r}{2}\right)$$

i.e.,

$$\log^{[2]} T(r,f_o g) \geq \frac{r}{2} + O(1).$$

Combining the above two inequalities, we have

$$\frac{\log^{[2]} T(r,f_o g)}{\log T(r,G)} \geq \frac{\frac{r}{2}+O(1)}{\log r + O(1)}.$$

Therefore

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,f_o g)}{\log T(r,G)} = \infty,$$

which is contrary to Theorem 2.

Theorem 3 *Let f and g be any two entire functions such that $\rho_g < \lambda_f \leq \rho_f < \infty$ and for $n \geq 1, F = f^n Q[f]$ and $G = g^n Q[g]$. Also there exist entire functions a_i $(i = 1, 2,\ldots, n; n \leq \infty)$ with*

$$(i) T(r,a_i) = o\{T(r,g)\} \text{ as } r\to\infty \text{ for } i = 1,2,\ldots,n$$

and

$$(ii) \sum_{i=1}^{n} \delta(a_i;g) = 1.$$

Then

$$\lim_{r\to\infty} \frac{\{\log T(r,f_o g)\}^2}{T(r,F)T(r,G)} = 0.$$

Proof. In view of the inequality

$$T(r,g) \leq \log^+ M(r,g)$$

and Lemma 1, we obtain for all sufficiently large values of r that

$$T(r,f_o g) \leq \{1+o(1)\}T(M(r,g),f)$$

i.e.,

$$\log T(r,f_o g) \leq \log\{1+o(1)\} + \log T(M(r,g),f)$$

i.e.,

$$\log T(r,f_o g) \leq o(1) + (\rho_f+\varepsilon)\log M(r,g)$$

i.e.,

$$\log T(r,f_o g) \leq o(1) + (\rho_f+\varepsilon)r^{(\rho_g+\varepsilon)}. \tag{11}$$

Again in view of Lemma 3, we get for all sufficiently large values of r that

$$\log T(r,F) > (\lambda_F - \varepsilon)\log r$$

i.e.,

$$\log T(r,F) > (\lambda_f - \varepsilon)\log r$$

i.e.,

$$T(r,F) > r^{\lambda_f - \varepsilon}. \tag{12}$$

Now combining (11) and (12), it follows for all sufficiently large values of r that

$$\frac{\log T(r,f_o g)}{T(r,F)} \leq \frac{o(1)+(\rho_f+\varepsilon)r^{(\rho_g+\varepsilon)}}{r^{\lambda_f-\varepsilon}}. \tag{13}$$

Since $\rho_g < \lambda_f$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g + \varepsilon < \lambda_f - \varepsilon. \tag{14}$$

So in view of (13) and (14), it follows that

$$\lim_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,F)} = 0. \tag{15}$$

Again from Lemma 4 and Lemma 8, we get for all sufficiently large values of r that

$$\frac{\log T(r,f_o g)}{T(r,G)} \leq \frac{o(1)+(\rho_f+\varepsilon)\log M(r,g)}{T(r,G)}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log T(r,f_o g)}{T(r,G)} \leq (\rho_f+\varepsilon)\limsup_{r\to\infty} \frac{\log M(r,g)}{T(r,G)}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log T(r,f_o g)}{T(r,G)} \le (\rho_f+\varepsilon)\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,G)}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log T(r,f_o g)}{T(r,G)} \le (\rho_f+\varepsilon).\pi.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log T(r,f_o g)}{T(r,G)} \le \rho_f.\pi. \qquad (16)$$

Combining (15) and (16), we obtain that

$$\limsup_{r\to\infty}\frac{\{\log T(r,f_o g)\}^2}{T(r,F)T(r,G)}$$

$$=\lim_{r\to\infty}\frac{\log T(r,f_o g)}{T(r,F)}.\limsup_{r\to\infty}\frac{\log T(r,f_o g)}{T(r,G)}$$

$$\le 0.\pi.\rho_f=0,$$

i.e.,

$$\lim_{r\to\infty}\frac{\{\log T(r,f_o g)\}^2}{T(r,F)T(r,G)}=0.$$

This proves the theorem.

Theorem 4 *Let f and g be any two entire functions satisfying the following conditions:* (i) $\lambda_f>0$ (ii) $\overline{\rho}_f<\infty$ (iii) $0<\lambda_g\le\rho_g$ *and also let for* $n\ge 1, F=f^nQ[f]$. *Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f_o g)}{\log^{[2]}T(r,F)} \ge \max\{\frac{\lambda_g}{\overline{\lambda}_f},\frac{\rho_g}{\overline{\rho}_f}\}.$$

Proof. We know that for $r>0$ {cf. [8]} and for all sufficiently large values of r,

$$T(r,f_o g)\ge\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\}. \qquad (17)$$

Since λ_f and λ_g are the lower orders of f and g respectively then for given $\varepsilon(>0)$ and for all sufficiently large values of r we obtain that

$$\log M(r,f)>r^{\lambda_f-\varepsilon}$$

and

$$\log M(r,g)>r^{\lambda_g-\varepsilon}$$

where $0<\varepsilon<\min\{\lambda_f,\lambda_g\}$. So from (17) we have for all sufficiently large values of r,

$$T(r,f_o g)\ge\frac{1}{3}\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1)\right\}^{\lambda_f-\varepsilon}$$

i.e.,

$$T(r,f_o g)\ge\frac{1}{3}\{\frac{1}{9}M(\frac{r}{4},g)\}^{\lambda_f-\varepsilon}$$

i.e.,

$$\log T(r,f_o g)\ge O(1)+(\lambda_f-\varepsilon)\log M(\frac{r}{4},g)$$

i.e.,

$$\log T(r,f_o g)\ge O(1)+(\lambda_f-\varepsilon)(\frac{r}{4})^{\lambda_g-\varepsilon}$$

i.e.,

$$\log^{[2]}T(r,f_o g)\ge O(1)+(\lambda_g-\varepsilon)\log r. \qquad (18)$$

Again in view of Lemma 1, we get for a sequence of values r tending to infinity that

$$\log^{[2]}T(r,F)\le(\overline{\lambda}_F+\varepsilon)\log r$$

i.e.,

$$\log^{[2]}T(r,F)\le(\overline{\lambda}_f+\varepsilon)\log r. \qquad (19)$$

Combining (18) and (19), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]}T(r,f_o g)}{\log^{[2]}T(r,F)}\ge\frac{O(1)+(\lambda_g-\varepsilon)\log r}{(\overline{\lambda}_f+\varepsilon)\log r}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f_o g)}{\log^{[2]}T(r,F)}\ge\frac{\lambda_g}{\overline{\lambda}_f}. \qquad (20)$$

Again from (17), we get for a sequence of values of r tending to infinity that

$$\log T(r, f_o g) \geq O(1) + (\lambda_f - \varepsilon)(\frac{r}{4})^{\rho_g - \varepsilon}$$

i.e.,

$$\log^{[2]} T(r, f_o g) \geq O(1) + \left(\rho_g - \varepsilon\right) \log r. \qquad (21)$$

Also in view of Lemma 5, for all sufficiently large values of r we have

$$\log^{[2]} T(r, F) \leq \left(\overline{\rho}_F + \varepsilon\right) \log r$$

i.e.,

$$\log^{[2]} T(r, F) \leq \left(\overline{\rho}_f + \varepsilon\right) \log r. \qquad (22)$$

Now from (21) and (22), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \geq \frac{O(1) + \left(\rho_g - \varepsilon\right) \log r}{\left(\overline{\rho}_f + \varepsilon\right) \log r}.$$

As $\varepsilon (0 < \varepsilon < \rho_g)$ is arbitrary, we obtain from above that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \geq \frac{\rho_g}{\overline{\rho}_f}. \qquad (23)$$

Therefore from (20) and (23), we get that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \geq \max\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

Thus the theorem is established.

Theorem 5 *Let f be meromorphic and g be entire such that $(i)\ 0 < \overline{\lambda}_f < \overline{\rho}_f$, $(ii)\ \rho_g < \infty$, $(iii)\ \rho_f < \infty$ and (iv) for $n \geq 1$, $F = f^n Q[f]$. Then*

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \leq \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

Proof. In view of Lemma 1 and the inequality

$$T(r, g) \leq \log^+ M(r, g),$$

we obtain for all sufficiently large values of r that

$$\log T(r, f_o g) \leq o(1) + \left(\rho_f + \varepsilon\right) \log M(r, g). \qquad (24)$$

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) \leq r^{\lambda_g + \varepsilon}. \qquad (25)$$

Combining (24) and (25), it follows for a sequence of values of r tending to infinity that

$$\log T(r, f_o g) \leq o(1) + (\rho_f + \varepsilon) r^{\lambda_g + \varepsilon}$$

i.e.,

$$\log T(r, f_o g) \leq r^{\lambda_g + \varepsilon} \{o(1) + (\rho_f + \varepsilon)\}$$

i.e.,

$$\log^{[2]} T(r, f_o g) \leq O(1) + \left(\lambda_g + \varepsilon\right) \log r. \qquad (26)$$

Again in view of Lemma 5, we obtain for all sufficiently large values of r that

$$\log^{[2]} T(r, F) > \left(\overline{\lambda}_F - \varepsilon\right) \log r$$

i.e.,

$$\log^{[2]} T(r, F) > \left(\overline{\lambda}_f - \varepsilon\right) \log r. \qquad (27)$$

Now from (26) and (27), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \leq \frac{O(1) + \left(\lambda_g + \varepsilon\right) \log r}{\left(\overline{\lambda}_f - \varepsilon\right) \log r}.$$

As $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \leq \frac{\lambda_g}{\overline{\lambda}_f}. \qquad (28)$$

In view of Lemma 1, we obtain for all sufficiently large values of r that

$$\log^{[2]} T(r, f_o g) \leq O(1) + \left(\rho_g + \varepsilon\right) \log r. \qquad (29)$$

Also by Remark 1, it follows for a sequence of values of r tending to infinity that

$$\log^{[2]} T(r, F) > \left(\overline{\rho}_F - \varepsilon\right) \log r$$

i.e.,

$$\log^{[2]} T(r, F) > \left(\overline{\rho}_f - \varepsilon\right) \log r. \qquad (30)$$

Combining (29) and (30), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \le \frac{O(1) + (\rho_g + \varepsilon)\log r}{(\overline{\rho}_f - \varepsilon)\log r}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \le \frac{\rho_g}{\overline{\rho}_f}. \qquad (31)$$

Now from (28) and (31), we get that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \le \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\}.$$

This proves the theorem.

The following theorem is a natural consequence of Theorem 4 and Theorem 5:

Theorem 6 *Let* f *and* g *be any two entire functions such that*
$(i)\ 0 < \overline{\lambda}_f < \overline{\rho}_f < \infty, (ii)\ 0 < \lambda_f \le \rho_f < \infty, (iii)\ 0 < \lambda_g \le \rho_g < \infty$ *and*
$(iv) for\ n \ge 1, F = f^n Q[f]$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)} \le \min\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\} \le \max\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\} \le \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r, F)}.$$

Theorem 7 *Let* f *be meromorphic and* g *be entire satisfying* $0 < \lambda_f \le \rho_f < \infty, \rho_g > 0$ *and also let for* $n \ge 1$, $F = f^n Q[f]$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log^{[2]} T(\exp(r^{\mu}), F)} = \infty,$$

where $0 < \mu < \rho_g$.

Proof. Let $0 < \mu' < \rho_g$. Then in view of Lemma 2, we get for a sequence of values of r tending to infinity that

$$\log T(r, f_o g) \ge \log T(\exp(r^{\mu}), f)$$

i.e.,

$$\log T(r, f_o g) \ge (\lambda_f - \varepsilon)\log\{\exp(r^{\mu'})\}$$

i.e.,

$$\log T(r, f_o g) \ge (\lambda_f - \varepsilon) r^{\mu'}$$

i.e.,

$$\log^{[2]} T(r, f_o g) \ge O(1) + \mu' \log r. \qquad (32)$$

Again in view of Lemma 3, we have for all sufficiently large values of r,

$$\log T(\exp(r^{\mu}), F) \le (\rho_F + \varepsilon)\log\{\exp(r^{\mu})\}$$

i.e.,

$$\log T(\exp(r^{\mu}), F) \le (\rho_f + \varepsilon) r^{\mu}$$

i.e.,

$$\log T(\exp(r^{\mu}), F) \le O(1) + \mu \log r. \qquad (33)$$

Now combining (32) and (33), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log^{[2]} T(\exp(r^{\mu}), F)} \ge \frac{O(1) + \mu' r^{\rho_g}}{O(1) + \mu \log r}$$

from which the theorem follows.

Remark 2 *The condition* $\rho_g > 0$ *is necessary in Theorem 7 as we see in the following example.*

Example 2 *Let* $f = \exp z, g = z$ *and* $\mu = 1(> 0)$. *Then* $f_o g = \exp z$ *and for* $n \ge 1, F = f^n Q[f]$. *Taking* $n = 1, A_j = 1, n_{0j} = 1$ *and* $n_{1j} = \cdots = n_{kj} = 0$; *we see that* $F = \exp 2z$. *Then*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} = 1,$$

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} = 1$$

and

$$\rho_g = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, g)}{\log r} = 0.$$

Also we get that

$$T(r, f) = \frac{r}{\pi}.$$

Therefore

$$T(\exp(r^{\rho_g}), f_o g) = \frac{e}{\pi}$$

and

$$T(\exp(r^{\mu}), F) = \frac{2\exp r}{\pi}.$$

So from the above two expressions we obtain that

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log^{[2]} T(\exp(r^{\mu}), F)} = \frac{O(1)}{\log r + O(1)}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log^{[2]} T(\exp(r^{\mu}), F)} = 0,$$

which contradicts Theorem 7.

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