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Establishing Relations among Various Measures by Using Well Known Inequalities

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ABSTRACT: *In this paper, we are establishing many interesting and important relations among several divergence measures by using known inequalities. Actually this work is application of well known inequalities in information theory. Except various relations, we tried to get bounds of* $N_k^*(P,Q), J_k^*(P,Q), \Delta_k(P,Q), E_k^*(P,Q), S^*(P,Q), L(P,Q), \psi M(P,Q), R_2(P,Q)$ *in terms of standard divergence measures. Some relations in terms of Arithmetic Mean* $A(P,Q)$*, Geometric Mean* $G^*(P,Q)$*, Harmonic Mean* $H(P,Q)$*, Heronian Mean* $N(P,Q)$*, Contra Harmonic Mean* $C(P,Q)$*, Root Mean Square* $S(P,Q)$ *and Centroidal Mean* $R(P,Q)$*, are also obtained.*

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I. Introduction

Let $\Gamma_n = \left\{ P = (p_1, p_2, p_3 ...., p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3, ..., n$, then we have to suppose that $0 f(0) = 0 f\left(\frac{0}{0}\right) = 0$. **Csiszar's f- divergence [1]** is a generalized information divergence measure, which is given by:

$$C_f(P,Q) = \sum_{i=1}^{n} q_i \, f\left(\frac{p_i}{q_i}\right) \tag{1}$$

Where f: $(0,\infty) \rightarrow$ R (set of real no.) is a convex function and P, Q $\in \Gamma_n$. Many known divergences can be obtained from these generalized measures by suitably defining the convex function f.

By (1), we obtain the following divergence measures:

Following measures are due to **(Jain and Srivastava [7]).**

$$E_k^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{(p_i \, q_i)^{\frac{k}{2}}}, \ k = 1, 3, 5, 7, ... \tag{2}$$

$$J_k^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{(p_i \, q_i)^{\frac{k}{2}}} \exp\left\{ \frac{(p_i - q_i)^2}{p_i \, q_i} \right\}, \ k = 1, 3, 5, 7, ... \tag{3}$$

Following measures are due to **Kumar P. and others.**

$$S^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i + q_i)(p_i - q_i)^2}{p_i \, q_i} \log\left(\frac{p_i + q_i}{2\sqrt{p_i \, q_i}} \right) \quad \textbf{(Kumar P. and Chhina [9])} \tag{4}$$

$$\psi M(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^2}{2\left(p_i\, q_i\right)^{\frac{3}{2}}} \quad \textbf{(Kumar P. and Johnson [11])} \tag{5}$$

$$L(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2}{p_i+q_i}\log\left(\frac{p_i+q_i}{2\sqrt{p_i\, q_i}}\right) \quad \textbf{(Kumar P. and Hunter [10])} \tag{6}$$

Renyi's second order entropy (Renyi A [12]).

$$R_2(P,Q)=\sum_{i=1}^{n}\frac{p_i^2}{q_i} \tag{7}$$

Puri and Vineze Divergence Measures (Kafka,Osterreicher and Vincze [8]).

$$\Delta_k(P,Q)=\sum_{i=1}^{n}\frac{\left|p_i-q_i\right|^{k+1}}{\left(p_i+q_i\right)^k},\ k\in(0,\infty) \tag{8}$$

Relative Jensen- Shannon divergence (Sibson [13]).

$$F(P,Q)=\sum_{i=1}^{n}p_i\log\left(\frac{2\,p_i}{p_i+q_i}\right) \tag{9}$$

Relative Arithmetic- Geometric Divergence (Taneja [14]).

$$G(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\,p_i}\right) \tag{10}$$

Arithmetic- Geometric Mean divergence Measure (Taneja [14]).

$$T(P,Q)=\frac{1}{2}\left[G(P,Q)+G(Q,P)\right]=\sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2\sqrt{p_i\, q_i}}\right) \tag{11}$$

Where $G(P,Q)$ is given by (10).

Symmetric Chi- square Divergence (Dragomir, Sunde and Buse [4]).

$$\Psi(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2\left(p_i+q_i\right)}{p_i\, q_i} \tag{12}$$

Relative J- Divergence (Dragomir, Gluscevic and Pearce [3]).

$$J_R(P,Q)=2\left[F(Q,P)+G(Q,P)\right]=\sum_{i=1}^{n}\left(p_i-q_i\right)\log\left(\frac{p_i+q_i}{2q_i}\right) \tag{13}$$

Where $F(P,Q)\ and\ G(P,Q)$ are given by (9) and (10) respectively.

Hellinger Discrimination (Hellinger [5]).

$$h(P,Q)=\frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i}-\sqrt{q_i}\right)^2 \tag{14}$$

Triangular Discrimination (Dacunha- Castelle [2]).

$$\Delta(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2}{p_i+q_i} \tag{15}$$

Except above, we obtain the following divergence measures **(Due to Jain and Saraswat [6]).**

$$N_k^*(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^{2k}}{\left(p_i+q_i\right)^{2k-1}}\exp\left\{\frac{\left(p_i-q_i\right)^2}{\left(p_i+q_i\right)^2}\right\},\ k=1,2,3,\ldots \tag{16}$$

II. Well Known Inequalities

The following inequalities are famous in literature of pure and applied mathematics, which are important tools to prove many interesting and important results in information theory.

$$1+t \le e^t \le 1+t\,e^t, \quad t>0 \tag{17}$$

$$\frac{t}{1+t} \le \log(1+t) \le t, \quad t>0 \tag{18}$$

III. Relations Among Various Divergence Measures

Now, we shall obtain bounds of some measures in terms of other divergence measures and many important and interesting relations among several divergence measures by using inequalities (17) and (18) respectively.

Proposition 1: Let $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have the inequalities:

$$N_k^*(P,Q)-N_{k+1}^*(P,Q)\le\Delta_{2k-1}(P,Q) \tag{19}$$

And $$\Delta_{2k+1}(P,Q)\le N_{k+1}^*(P,Q) \tag{20}$$

Where $k=1,2,3,\ldots$, and $N_k^*(P,Q)$, $\Delta_k(P,Q)$ are given by (16) and (8) respectively.

Proof: Put $t=\frac{(p_i-q_i)^2}{(p_i+q_i)^2}$ in inequalities (17), we get

$$1+\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\le\exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\le1+\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2},$$

now multiply the above expression by $\frac{(p_i-q_i)^{2k}}{(p_i+q_i)^{2k-1}}$, $k=1,2,3,\ldots$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^n\frac{(p_i-q_i)^{2k}}{(p_i+q_i)^{2k-1}}+\sum_{i=1}^n\frac{(p_i-q_i)^{2k+2}}{(p_i+q_i)^{2k+1}}\le\sum_{i=1}^n\frac{(p_i-q_i)^{2k}}{(p_i+q_i)^{2k-1}}\exp\left\{\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\right\}$$

$$\le\sum_{i=1}^n\frac{(p_i-q_i)^{2k}}{(p_i+q_i)^{2k-1}}+\sum_{i=1}^n\frac{(p_i-q_i)^{2k+2}}{(p_i+q_i)^{2k+1}}\exp\left\{\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\right\}$$

i. e. $$\Delta_{2k-1}(P,Q)+\Delta_{2k+1}(P,Q)\le N_k^*(P,Q)\le\Delta_{2k-1}(P,Q)+N_{k+1}^*(P,Q) \tag{21}$$

From second and third part of (21), we get inequality (19) and from first and third part, we get (20).

Now at k=1, 2, 3 … we get the followings [from inequalities (19) and (20)]:

At k=1 $\Rightarrow$ $N_1^*(P,Q)-N_2^*(P,Q)\le\Delta(P,Q)$ *and* $\{\because \Delta_1(P,Q)=\Delta(P,Q)\}$ (22)

$\Delta_3(P,Q)\le N_2^*(P,Q)$

At k=2 $\Rightarrow$ $N_2^*(P,Q)-N_3^*(P,Q)\le\Delta_3(P,Q)$ *and*

$\Delta_5(P,Q)\le N_3^*(P,Q)$

At k=3 $\Rightarrow$ $N_3^*(P,Q)-N_4^*(P,Q)\le\Delta_5(P,Q)$ *and*

$\Delta_7(P,Q)\le N_4^*(P,Q)$ and so on…

Proposition 2: Let $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have the inequalities:

$$J_k^*(P,Q)-J_{k+2}^*(P,Q)\le E^*_k(P,Q) \tag{23}$$

And $$E^*_{k+2}(P,Q)\le J_{k+2}^*(P,Q) \tag{24}$$

Where $k=1,3,5,\ldots$, and $E_k^*(P,Q)$, $J^*_k(P,Q)$ are given by (2) and (3) respectively.

Proof: Put $t = \dfrac{(p_i - q_i)^2}{p_i\, q_i}$ in inequalities (17), we get

$$1 + \frac{(p_i - q_i)^2}{(p_i\, q_i)} \le \exp\frac{(p_i - q_i)^2}{(p_i\, q_i)} \le 1 + \frac{(p_i - q_i)^2}{(p_i\, q_i)} \exp\frac{(p_i - q_i)^2}{(p_i\, q_i)},$$

Now multiply the above expression by $\dfrac{(p_i - q_i)^{k+1}}{(p_i\, q_i)^{k/2}}$, $k = 1, 3, 5, \ldots$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{(p_i\, q_i)^{k/2}} + \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+3}}{(p_i\, q_i)^{\frac{k}{2}+1}} \le \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{(p_i\, q_i)^{k/2}} \exp\left\{\frac{(p_i - q_i)^2}{(p_i\, q_i)}\right\}$$

$$\le \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+1}}{(p_i\, q_i)^{k/2}} + \sum_{i=1}^{n} \frac{(p_i - q_i)^{k+3}}{(p_i\, q_i)^{\frac{k}{2}+1}} \exp\left\{\frac{(p_i - q_i)^2}{(p_i\, q_i)}\right\}$$

i. e. $$E_k^*(P,Q) + E_{k+2}^*(P,Q) \le J_k^*(P,Q) \le E_k^*(P,Q) + J_{k+2}^*(P,Q) \tag{25}$$

From second and third part of (25), we get inequality (23) and from first and third part, we get (24).

Now at k=1, 3, 5 … we get the followings [from inequalities (23) and (24)]:

At k=1 $\Rightarrow$ $J_1^*(P,Q) - J_3^*(P,Q) \le E_1^*(P,Q)$ *and*

$E^*_3(P,Q) \le J_3^*(P,Q)$

At k=3 $\Rightarrow$ $J_3^*(P,Q) - J_5^*(P,Q) \le E^*_3(P,Q)$ *and*

$E^*_5(P,Q) \le J_5^*(P,Q)$ and so on…

Except these, from first and second part of the inequalities (25), we can easily see that

$$E_1^*(P,Q) \le J_1^*(P,Q) \tag{26}$$

Proposition 3: Let $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have the inequalities:

$$\psi(P,Q) - 2E_1^*(P,Q) \le S^*(P,Q) \tag{27}$$

And $$S^*(P,Q) + \psi(P,Q) \le \psi M(P,Q) \tag{28}$$

Where $\psi(P,Q)$, $E_1^*(P,Q)$, $S^*(P,Q)$ *and* $\psi M(P,Q)$ are given by (12), (2), (4) and (5) respectively.

Proof: Put $t = \dfrac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}$ in inequalities (18), we get

$$\frac{\dfrac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}}{1 + \dfrac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}} \le \log\left(1 + \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}\right) \le \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}$$

i. e. $$\frac{p_i + q_i - 2\sqrt{p_i q_i}}{p_i + q_i} \le \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \le \frac{p_i + q_i - 2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}$$

Now multiply the above expression by $\dfrac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i}\le\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\log\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right)$$

$$\le\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}$$

i.e.

$$\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i+q_i)}{p_i\ q_i}-2\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{\sqrt{p_iq_i}}\le S^*(P,Q)\le\sum_{i=1}^{n}\frac{(p_i^2-q_i^2)^2}{2(p_i\ q_i)^{\frac{3}{2}}}-\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i+q_i)}{p_i\ q_i}$$

i. e. $$\psi(P,Q)-2E_1^*(P,Q)\le S^*(P,Q)\le\psi M(P,Q)-\psi(P,Q) \qquad (29)$$

From first and second part of (29), we get inequality (27) and from second and third part, we get (28).
Except these, if we add (27) and (28), we get the following

$$2\,\psi(P,Q)\le\psi M(P,Q)+2\,E_1^*(P,Q) \qquad (30)$$

From second and third part of the inequalities (29), we can easily see that

$$S^*(P,Q)\le\psi M(P,Q) \qquad (31)$$

By taking both (27) and (31), we can write

$$\psi(P,Q)-2E_1^*(P,Q)\le S^*(P,Q)\le\psi M(P,Q) \qquad (32)$$

Proposition 4: Let $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have the inequalities:

$$L(P,Q)+\Delta(P,Q)\le\frac{1}{2}E_1^*(P,Q) \qquad (33)$$

And $$\Delta(P,Q)\le L(P,Q)+2\sum_{i=1}^{n}\frac{\sqrt{p_iq_i}\,(p_i-q_i)^2}{(p_i+q_i)^2} \qquad (34)$$

Where $L(P,Q)$, $E_1^*(P,Q)$, $\Delta(P,Q)$ are given by (6), (2) and (15) respectively.

Proof: Put $t=\dfrac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_iq_i}}$ in inequalities (18), we get

$$\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i}\le\log\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right)\le\frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}$$

Now multiply the above expression by $\dfrac{(p_i-q_i)^2}{p_i+q_i}$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i}\le\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}\log\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right)\le\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}\frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}$$

i. e. $$\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}-2\sum_{i=1}^{n}\frac{\sqrt{p_iq_i}\,(p_i-q_i)^2}{(p_i+q_i)^2}\le L(P,Q)\le\frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{\sqrt{p_iq_i}}-\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}$$

i. e. $$\Delta(P,Q)-2\sum_{i=1}^{n}\frac{\sqrt{p_iq_i}\,(p_i-q_i)^2}{(p_i+q_i)^2}\le L(P,Q)\le\frac{1}{2}E_1^*(P,Q)-\Delta(P,Q) \qquad (35)$$

From second and third part of (35), we get inequality (33) and from first and second part, we get (34).
From inequality (33), we can easily see that

$$\Delta(P,Q) \le \frac{1}{2} E_1^*(P,Q) \tag{36}$$

Proposition 5: Let $(P,Q) \in \Gamma_n \times \Gamma_n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then we have the inequalities:

$$A(P,Q) \le h(P,Q) \le T(P,Q) \tag{37}$$

$$A(P,Q) + h(P,Q) \le \frac{1}{4} \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{\sqrt{p_i q_i}} \tag{38}$$

And
$$A(P,Q) + T(P,Q) \le \frac{1}{4} \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{\sqrt{p_i q_i}} \tag{39}$$

Where $T(P,Q), h(P,Q)$ are given by (11) and (14) respectively and $A(P,Q) = \sum_{i=1}^{n} \frac{p_i + q_i}{2} = 1$ is well known Arithmetic Mean Divergence.

Proof: Put $t = \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}$ in inequalities (18), we get

$$\frac{p_i + q_i - 2\sqrt{p_i q_i}}{p_i + q_i} \le \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \le \frac{p_i + q_i - 2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}$$

Now multiply the above expression by $\frac{p_i + q_i}{2}$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n} \frac{(p_i + q_i)}{2} \frac{p_i + q_i - 2\sqrt{p_i q_i}}{p_i + q_i} \le \sum_{i=1}^{n} \frac{(p_i + q_i)}{2} \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \le \sum_{i=1}^{n} \frac{(p_i + q_i)}{2} \frac{p_i + q_i - 2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}$$

i. e.
$$\sum_{i=1}^{n} \frac{p_i + q_i - 2\sqrt{p_i q_i}}{2} \le T(P,Q) \le \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{4\sqrt{p_i q_i}} - 1$$

i. e.
$$\sum_{i=1}^{n} \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2} \le T(P,Q) \le \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{4\sqrt{p_i q_i}} - 1$$

i. e.
$$h(P,Q) \le T(P,Q) \le \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{4\sqrt{p_i q_i}} - 1 \tag{40}$$

From first and third part of (40), we get inequality (38) and from second and third part, we get (39).
Except these, from (38) and (40), we can easily see the followings

$$A(P,Q) \le \frac{1}{4} \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{\sqrt{p_i q_i}} \tag{41}$$

$$h(P,Q) \le \frac{1}{4} \sum_{i=1}^{n} \frac{(p_i + q_i)^2}{\sqrt{p_i q_i}} \tag{42}$$

And
$$h(P,Q) \le T(P,Q) \tag{43}$$

Now do (41)-(42), we get

$$A(P,Q)-h(P,Q)\le 0 \Rightarrow A(P,Q)\le h(P,Q) \quad (44)$$

By taking both (43) and (44), we get the inequalities (37).

Proposition 6: Let $(P,Q)\in\Gamma_n\times\Gamma_n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then we have the inequalities:

$$G(Q,P)\ge \frac{1}{2}-\log 2 \quad (45)$$

And $$\log 2 + G(Q,P)\le \frac{1}{2}\left[R_2(P,Q)+1\right] \quad (46)$$

Where $R_2(P,Q)$, $G(Q,P)$ are given by (7) and (10) respectively.

Proof: Put $t=\frac{p_i}{q_i}$ in inequalities (18), we get

$$\frac{p_i}{p_i+q_i}\le \log\left(\frac{p_i+q_i}{q_i}\right)\le \frac{p_i}{q_i}$$

Now multiply the above expression by $\frac{p_i+q_i}{2}$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n}\frac{p_i+q_i}{2}\frac{p_i}{p_i+q_i}\le \sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\left(\frac{2}{2}\frac{p_i+q_i}{q_i}\right)\le \sum_{i=1}^{n}\frac{p_i+q_i}{2}\frac{p_i}{q_i}$$

i. e. $$\sum_{i=1}^{n}\frac{p_i}{2}\le \log 2\sum_{i=1}^{n}\frac{p_i+q_i}{2}+\sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2q_i}\right)\le \sum_{i=1}^{n}\frac{p_i^{\,2}}{2q_i}+\sum_{i=1}^{n}\frac{p_i}{2}$$

i. e. $$\frac{1}{2}\le \log 2 + G(Q,P)\le \frac{1}{2}\left[R_2(P,Q)+1\right] \quad (47)$$

From first and second part of (47), we get inequality (45) and from second and third part, we get (46).

Proposition 7: Let $(P,Q)\in\Gamma_n\times\Gamma_n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then we have the inequalities:

$$\log 2 - F(P,Q)\le A(P,Q) \quad (48)$$

And $$\frac{1}{2}H(P,Q)+F(P,Q)\le \log 2 \quad (49)$$

Where $F(P,Q)$ is given by (9), $A(P,Q)=\sum_{i=1}^{n}\frac{p_i+q_i}{2}=1\ and\ H(P,Q)=\sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i}$ are Arithmetic Mean and Harmonic Mean Divergences respectively.

Proof: Put $t=\frac{p_i}{q_i}$ in inequalities (18), we get

$$\frac{p_i}{p_i+q_i}\le \log\left(\frac{p_i+q_i}{q_i}\right)\le \frac{p_i}{q_i}$$

Now multiply the above expression by $2q_i$ and sum over all i=1, 2, 3…n, we get

$$\sum_{i=1}^{n}2q_i\frac{p_i}{p_i+q_i}\le \sum_{i=1}^{n}2q_i\log\left(\frac{2}{2}\frac{p_i+q_i}{q_i}\right)\le \sum_{i=1}^{n}2q_i\frac{p_i}{q_i}$$

i. e. $$H(P,Q)\le 2\log 2\sum_{i=1}^{n}q_i-2\sum_{i=1}^{n}q_i\log\left(\frac{2q_i}{p_i+q_i}\right)\le 2\sum_{i=1}^{n}p_i$$

i. e. $H(P,Q) \le 2\log 2 - 2F(Q,P) \le 2$

After interchanging P and Q, we get the following

$$H(P,Q) \le 2\log 2 - 2F(P,Q) \le 2 \tag{50}$$

from second and third part of (50), we get inequality (48) and from first and second part, we get (49).

Some Relations:

$\because H(P,Q) \le G^*(P,Q) \le N(P,Q) \le A(P,Q) \le R(P,Q) \le S(P,Q) \le C(P,Q)$ **(Taneja [15]).** (51)

The above inequalities (51) is a famous relation among seven means, where $H(P,Q), G^*(P,Q), N(P,Q), A(P,Q), R(P,Q), S(P,Q), C(P,Q)$ are mentioned in abstract.

Now we can get some other important relations among various divergences with the help of above inequalities, these are as follows.

- from (37) and (51), we get

$$H(P,Q) \le G^*(P,Q) \le N(P,Q) \le A(P,Q) \le h(P,Q) \le T(P,Q) \tag{52}$$

- from (48) and (51), we get

$$\log 2 - F(P,Q) \le A(P,Q) \le R(P,Q) \le S(P,Q) \le C(P,Q) \tag{53}$$

- from (37) and (48), we get

$$\log 2 - F(P,Q) \le A(P,Q) \le h(P,Q) \le T(P,Q) \tag{54}$$

- do (46) - (48), we get

$$G(Q,P) + F(Q,P) \le \frac{1}{2}\left[R_2(P,Q) + 1\right] - A(P,Q)$$

i. e. $2A(P,Q) + 2\left[G(Q,P) + F(Q,P)\right] \le R_2(P,Q) + 1$

i. e. $2A(P,Q) + J_R(P,Q) \le R_2(P,Q) + 1$ (55)

- from (22), (26) and (36), we get

$$N_1^*(P,Q) - N_2^*(P,Q) \le \Delta(P,Q) \le \frac{1}{2}E_1^*(P,Q) \le \frac{1}{2}J_1^*(P,Q) \tag{56}$$

- from (22) and (23), we get

$$N_1^*(P,Q) - N_2^*(P,Q) \le \Delta(P,Q) \le \frac{1}{2}E_1^*(P,Q) - L(P,Q) \tag{57}$$

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