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The Bloch Space of Analytic functions

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Abstract: *We shall state and prove a characterization for the Bloch space and obtain analogous characterization for the little Bloch space of analytic functions on the unit disk in the complex plane. We shall also state and prove three containment results related to Bloch space and Little Bloch space.*
Keywords: *Bloch Space, Analytic Functions, Mobius Transformation*

I. Introduction

We let $= \{ z \in C / |z| < 1 \}$ D

For w∈D, the Mobius transformation ϕ_w is defined by

$$\phi_w(z) = \frac{w-z}{1-\overline{w}z} \quad for \ \ z \in D$$

Then

$$1-|\phi_w(z)|^2 = 1-\phi_w(z).\overline{\phi_w(z)}$$

$$= 1-\left(\frac{w-z}{1-\overline{\omega}z}\right)\left(\frac{\overline{w}-\overline{z}}{1-w\overline{z}}\right)$$

$$1-|\phi_w(z)|^2 = \frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{\left|1-\overline{w}z\right|^2} \qquad -(1)$$

So, the function ϕ_w maps D on to itself and ∂D on to itself. It is easy to verify that ϕ_w is its own inverse. Noting that $\phi_w^1(z) = \dfrac{\left(|w|^2-1\right)}{\left(\left|1-\overline{w}z\right|\right)^2}$, the above identity states:

$$\left(1-|z|^2\right)\left|\phi_w^1(z)\right| = 1-|\phi_w(z)|^2 \qquad -(2)$$

Bloch space B is the space of all analytic functions f on D for which

$$\sup_{z\in D} \left(1-|z|^2\right) \left| f^1(z)\right| < \infty$$

and B becomes a Banach space with respect to the semi norm

$$\|f\|_B = \sup_{z\in D}\left(1-|z|^2\right)\left|f^1(z)\right|$$

Using (2), we have

$$\begin{aligned}
\| fo\phi_w \|_{B} &= \underset{z\in D}{Sup}\left(1-|z|^2\right)\left|\left(fo\phi_w\right)^1(z)\right| \\
&= \underset{z\in D}{Sup}\left(1-|z|^2\right)\left|f^1(\phi_w(z))\right|\left|\phi_w^1(z)\right| \\
&= \underset{\phi_w(z)\in D}{Sup}\left(1-|\phi_w(z)|^2\right)\left|f^1(\phi_w(z))\right| \\
&= \| f \|_{B}
\end{aligned}$$

$$\therefore \| fo\phi_w \|_{B} = \| f \|_{B} \qquad - \quad (3)$$

Thus Bloch space is a Mobius invariant space.

In the next section, we shall state and prove a criterion for containment in the Bloch space and little Bloch space.

II. CHARACTERIZATION FOR BLOCH AND LITTLE BLOCH SPACE

A. THEOREM 1

For an analytic function f on D

$$f\in B \Leftrightarrow Sup\left\{\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{\left|1-\overline{w}z\right|}\left|\frac{f(z)-f(w)}{z-w}\right| : z,w\in D,\ z\neq w\right\}<\infty$$

Proof : Suppose for an analytic function f on D

$$Sup\left\{\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{\left|1-\overline{w}z\right|}\left|\frac{f(z)-f(w)}{z-w}\right| : z,w\in D,\ z\neq w\right\}<\infty$$

$$\Rightarrow \quad \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{\left|1-\overline{w}z\right|}\left|\frac{f(z)-f(w)}{z-w}\right|<\infty,\ \ \forall\, z,w\in D, z\neq w$$

Taking limit as w→z, we get

$$Sup\left(\frac{\left(1-|z|^2\right)^2\left|f^1(z)\right|}{\left(1-|z|^2\right)}\right)<\infty$$

$$\Rightarrow \underset{z\in D}{Sup}\left(1-|z|^2\right)\left|f^1(z)\right|<\infty$$

$$\Rightarrow f\in B$$

For the next part, suppose $f\in B$

$$\Rightarrow \underset{z\in D}{Sup}\left(1-|z|^2\right)\left|f^1(z)\right|<\infty$$

$$\Rightarrow \left(1-|z|^2\right)\left|f^1(z)\right|\leq \| f \|_{B},\ \forall\, z\in D \qquad - \quad (4)$$

Then for each $u \in D$, we have

$$f(u) - f(0) = \int_0^1 f^1(tu)\,u\,dt$$

$$\Rightarrow \ |f(u) - f(0)| = \left|\int_0^1 f^1(tu)\,u\,dt\right|$$

$$\leq \int_0^1 |f^1(tu)||u|\,dt$$

$$\leq \int_0^1 \frac{\|f\|_B}{1 - t^2|u|^2}|u|\,dt \qquad (\because (4))$$

$$\leq \int_0^1 \frac{\|f\|_B |u|}{1 - t|u|}\,dt$$

$$\therefore\ |f(u) - f(0)| < \|f\|_B \int_0^1 \frac{|u|}{1 - t|u|}\,dt = \|f\|_B |u| \frac{\left(\log(1 - t|u|)\right)_0^1}{-|u|}$$

$$= \|f\|_B \log(1 - |u|)^{-1} = \|f\|_B \log\left|\frac{1}{1 - |u|}\right|$$

$$< \|f\|_B \log\left|\frac{1 + |u|}{1 - |u|^2}\right|$$

$$\leq \|f\|_B \left(\frac{1 + |u|}{1 - |u|^2} - 1\right) \qquad (\therefore \log x \leq x - 1,\ x > 0)$$

$$\leq \|f\|_B \left(\frac{1 + |u| - 1 + |u|^2}{1 - |u|^2}\right)$$

$$\leq \|f\|_B \left(\frac{1 + |u| - 1 + |u|}{1 - |u|^2}\right)$$

$$\therefore\ |f(u) - f(0)| \leq \|f\|_B \frac{2|u|}{1 - |u|^2}, \quad \forall\ u \in D$$

Now for z, $w \in D$ replace f in the above inequality by $fo\phi_w$ and let u = ϕ_w(z). Using

$\phi_w(\phi_w(z)) = z$ and identities (1) and (3) we have

$$|(fo\phi_w)(u) - (fo\phi_w)(0)| \leq \|fo\phi_w\|_B \cdot \frac{2|\phi_w(z)|}{1 - |\phi_w(z)|^2}$$

$$\Rightarrow \left| f(z) - f(w) \right| \le \frac{\| f \|_B \dfrac{2|w-z|}{|1-\overline{w}z|}}{\dfrac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-\overline{w}z|^2}}$$

We briefly discuss the little Bloch space B0. The set of all analytic functions f on D for which

$$\lim_{|z|\to 1^-} \left(1-|z|^2\right)\left| f^1(z) \right| = 0$$

For an analytic function f on D and 0 < t < 1 the dilate ft is the function defined by $f_t(z) = f(tz)$. It is known that for an analytic function *f* on D:

$$f \in B_0 \quad iff \quad \| f - f_t \|_B \to 0 \quad as\ t \to \bar{1}$$

$$\therefore \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{w}z|} \left| \frac{f(z)-f(w)}{z-w} \right| \le 2\| f \|_B \qquad - \ (5)$$

$$\therefore Sup \left\{ \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{w}z|} \left| \frac{f(z)-f(w)}{z-w} \right| : z, w \in D,\ z \ne w \right\} \le 2\| f \|_B < \infty$$

In analogy to theorem (1), we have the following result.

B. THEOREM 2

For an analytic function f on D

$$\lim_{|z|\to 1^-} Sup \left\{ \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\overline{w}z|} \left| \frac{f(z)-f(w)}{z-w} \right| : z, w \in D,\ z \ne w \right\} = 0$$

Proof:

Taking limit as w→z in the condition of the statement, we get

$$\lim_{|z|\to 1^-} Sup \left\{ \frac{\left(1-|z|^2\right)^2}{\left|1-|z|^2\right|} \left| f^1(z) \right| \right\} = 0$$

$$\therefore \lim_{|z|\to 1^-} \left(1-|z|^2\right)\left| f^1(z) \right| = 0$$

$$\Rightarrow f \in B_0$$

Suppose f $\in B_0$, then $f - f_t \in B$

Applying inequality (5) for $f_t \in B$, we have

$$\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z|}\left|\frac{f_t(z)-f_t(w)}{z-w}\right| \le \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|z-w||1-\bar{w}z|}\cdot\frac{2\|f_t\|_B\, t|z-w||1-t^2\bar{w}z|}{\left(1-t^2|z|^2\right)\left(1-t^2|w|^2\right)}$$

$$=\frac{2t}{\left(1-t^2\right)^2}\|f\|_B\left(1-|z|^2\right) \quad - \quad (6)$$

Applying inequality (5) for f - $f_t \in$B, we have

$$\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z|}\left|\frac{(f-f_t)(z)-(f-f_t)(w)}{z-w}\right| \le 2\|f-f_t\|_B \quad - \ (7)$$

Inequality (6), (7) and triangle inequality imply that

$$\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z|}\left|\frac{f(z)-f(w)}{z-w}\right|$$

$$=\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z||z-w|}\left|(f-f_t)(z)-(f-f_t)(w)+f_t(z)-f_t(w)\right|$$

$$\le \frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z||z-w|}\left|(f-f_t)(z)-(f-f_t)(w)\right|$$

$$+\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z||z-w|}\left|f_t(z)-f_t(w)\right|$$

$$\le 2\|f-f_t\|_B+\frac{2t}{\left(1-t^2\right)^2}\|f\|_B\left(1-|z|^2\right)$$

Now first letting $|z| \to \bar{1}$ and then $t \to \bar{1}$, we get

$$\lim_{|z|\to 1^-} Sup\left\{\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z|}\left|\frac{f(z)-f(w)}{z-w}\right| : z,w\in D,\ z\ne w\right\}=0$$

In the next section, we shall prove three results related to containment of Bloch and Little Bloch space

III. CONTAINMENT RESULTS OF BLOCH AND LITTLE BLOCH SPACE

Let ϕ be a bounded analytic function on D, then there exists a constant

$$M>0 such that \left|\phi(z)\right|\le M, \forall z\in D$$

From Cauchy's integral formula, we have

$$\phi'(z)=\frac{1}{2\pi i}\int_C\frac{\phi(w)\,dw}{(w-z)^2}$$

where C is any closed disc of radius r in neighbourhood of 1 and containing z, then $\left|\phi'(z)\right| \le \frac{4M}{r}$ in the concentric disc of radius $\frac{r}{2}$. This implies that $\phi'(z)$ is bounded in any neighbourhood of 1 contained in D whenever so is $\phi(z)$.

A. THEOREM 3

If $f \in B$ then $f + k \in B$ where $k \in C$ is a constant

Proof: It is very easy to see that

$$f'(z) = (f+k)'(z)$$

Therefore $\sup_{z\in D}\left(1-|z|^2\right)\left|f'(z)\right| = \sup_{z\in D}\left(1-|z|^2\right)\left|(f+k)'(z)\right|$

Hence $f + k \in B$ whenever $f \in B$

B. THEOREM 4

If $f \in B_0$ is bounded and ϕ is any bounded and analytic function on D then $\phi f \in B_0$.

Proof: $f \in B_0 \Rightarrow \lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|f'(z)\right| = 0$

$$(\phi f)'(z) = \phi(z) f'(z) + f(z)\phi'(z)$$

Note that $\Rightarrow \left|(\phi f)'(z)\right| \le |\phi(z)|\left|f'(z)\right| + |f(z)|\left|\phi'(z)\right|$

$$\Rightarrow \left(1-|z|^2\right)\left|(\phi f)'(z)\right| \le \left(1-|z|^2\right)|\phi(z)|\left|f'(z)\right| + \left(1-|z|^2\right)|f(z)|\left|\phi'(z)\right|$$

Taking limit as $|z| \to 1^-$, the first term on RHS tends to 0 because of the hypothesis and ϕ is bounded and the second term since f and ϕ' bounded in the neighbourhood of 1 as ϕ is bounded on D tends to 0.

Hence $\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|(\phi f)'(z)\right| = 0$

Therefore $\phi f \in B_0$.

C. THEOREM 5

If f, g are bounded functions of B_0, then $fg \in B_0$.

Proof: From the definition of B_0,

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|f'(z)\right| = 0$$

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|g'(z)\right| = 0$$

Note that $0 < \left(1-|z|^2\right)\left|(fg)'(z)\right| \le \left(1-|z|^2\right)\left|g'(z)\right||f(z)| + \left(1-|z|^2\right)\left|f'(z)\right||g(z)|$

Taking limit as $|z| \to 1^-$, we get

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|(fg)'(z)\right| = 0$$

$$\therefore fg \in B_0.$$

IV. CONCLUSION

I invite interested readers to pursue geometric interpretation of characterization theorems that we proved in this paper and also similar containment results related to the Bloch space.

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