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About the 2-Banach Spaces

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Abstract: *In this paper are proved a few properties about convergent sequences into a real* 2*-normed space* $(L,\|\cdot,\cdot\|)$ *and into a* 2*-pre-Hilbert space* $(L,(\cdot,\cdot|\cdot))$, *which are actually generalization of appropriate properties of convergent sequences into a pre-Hilbert space. Also, are given two characterization of a* 2*-Banach spaces. These characterization in fact are generalization of appropriate results in Banach spaces.*

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I. Introduction

Let L be a real vector space with dimension greater than 1 and, $\|\cdot,\cdot\|$ be a real function defined on $L\times L$ which satisfies the following:

a) $\|x,y\|\geq 0$, for each $x,y\in L$ и $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent,

b) $\|x,y\|=\|y,x\|$, for each $x,y\in L$,

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for each $x,y\in L$ and for each $\alpha\in\mathbf{R}$,

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for each $x,y,z\in L$.

Function $\|\cdot,\cdot\|$ is called as 2-*norm of L*, and $(L,\|\cdot,\cdot\|)$ is called as *vector* 2-*normed space* ([1]).

Let $n>1$ be a natural number, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be real function on $L\times L\times L$ such that:

i) $(x,x|y)\geq 0$, for each $x,y\in L$ и $(x,x|y)=0$ if and only if a and b are linearly dependent,

ii) $(x,y|z)=(y,x|z)$, for each $x,y,z\in L$,

iii) $(x,x|y)=(y,y|x)$, for each $x,y\in L$,

iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for each $x,y,z\in L$ and for each $\alpha\in\mathbf{R}$, and

v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for each $x_1,x,y,z\in L$.

Function $(\cdot,\cdot|\cdot)$ is called as 2-*inner product*, and $(L,(\cdot,\cdot|\cdot))$ is called as 2-*pre-Hilbert space* ([2]).

Concepts of a 2-norm and a 2-inner product are two dimensional analogies of concepts of a norm and an inner product. R. Ehret ([3]) proved that, if $(L,(\cdot,\cdot|\cdot))$ be 2-pre-Hilbert space, than

$$\|x,y\|=(x,x|y)^{1/2},\ \text{for each}\ x,y\in L \qquad (1)$$

defines 2-norm. So, we get 2-normed vector space $(L,\|\cdot,\cdot\|)$.

II. Convergent And Cauchy Sequences In 2-Pre-Hilbert Space

The term convergent sequence in 2-normed space is given by A. White, and he also proved a few properties according to this term. The sequence $\{x_n\}_{n=1}^{\infty}$ into vector 2-normed space is called as *convergent* if there exists $x\in L$ such that $\lim\limits_{n\to\infty}\|x_n-x,y\|=0$, for every $y\in L$. Vector $x\in L$ is called as *bound* of the sequence $\{x_n\}_{n=1}^{\infty}$ and we note $\lim\limits_{n\to\infty}x_n=x$ or $x_n\to x$, $n\to\infty$, ([4]).

Theorem 1 ([4]). Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space over L.

a) If $\lim\limits_{n\to\infty}x_n=x$, $\lim\limits_{n\to\infty}y_n=y$, $\lim\limits_{n\to\infty}\alpha_n=\alpha$ and $\lim\limits_{n\to\infty}\beta_n=\beta$, then

$$\lim_{n\to\infty}(\alpha_n x_n + \beta_n y_n) = \alpha x + \beta y .$$

б) If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$. ■

Theorem 2 ([4]). a) Let L be a vector 2-normed space, $\{x_n\}_{n=1}^{\infty}$ be the sequence in L and $y \in L$ is such that $\lim_{n\to\infty} \| x_n - x_m, y \| = 0$. Then the real sequence $\{\| x_n - x, y \|\}_{n=1}^{\infty}$ is convergent for every $x \in L$.

b) Let L be a vector 2-normed space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in L and $y \in L$ is such that $\lim_{n\to\infty} \| x_n - x, y \| = 0$. Then $\lim_{n\to\infty} \| x_n, y \| = \| x, y \|$. ■

In [7] H. Gunawan was focused on convergent sequences in 2-pre-Hilbert space $(L,(\cdot,\cdot|\cdot))$, and he gave the term weakly convergent sequence in 2-pre-Hilbert space. According to this he proved a few corollaries.

Definition 1 ([5]). The sequence $\{x_n\}_{n=1}^{\infty}$ in the 2-pre-Hilbert space $(L,(\cdot,\cdot|\cdot))$ is called as *weakly convergent* if there exists $x \in L$ such that $\lim_{n\to\infty}(x_n - x, y | z) = 0$, for every $y, z \in L$. The vector $x \in L$ is called as *weak limit* of the sequence $\{x_n\}_{n=1}^{\infty}$ and is denoted as $x_n \xrightarrow{w} x$, $n \to \infty$.

Theorem 3. Let $(L,(\cdot,\cdot|\cdot))$ be a 2-pre-Hilbert space. If $x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$, $\alpha_n \to \alpha$ and $\beta_n \to \beta$, for $n \to \infty$, then $\alpha_n x_n + \beta_n y_n \xrightarrow{w} \alpha x + \beta y$, for $n \to \infty$.

Proof. The validity of this theorem is as a direct implication of Definition 1, the fact that every convergent real sequence is bounded and the following equalities:

$$\begin{aligned}\lim_{n\to\infty}(\alpha_n x_n + \beta_n y_n - \alpha x - \beta y, u | v) &= \lim_{n\to\infty}(\alpha_n x_n - \alpha x, u | v) + \lim_{n\to\infty}(\beta_n y_n - \beta y, u | v)\\ &= \lim_{n\to\infty}\alpha_n(x_n - x, u | v) + \lim_{n\to\infty}(\alpha_n - \alpha)(x, u | v) +\\ &\quad + \lim_{n\to\infty}\beta_n(y_n - y, u | v) + \lim_{n\to\infty}(\beta_n - \beta)(y, u | v) .\end{aligned}$$ ■

Corollary 1 ([5]). Let $(L,(\cdot,\cdot|\cdot))$ be a 2-pre-Hilbert space. If $x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$, for $n \to \infty$ and $\alpha, \beta \in \mathbf{R}$, then $\alpha x_n + \beta y_n \xrightarrow{w} \alpha x + \beta y$, for $n \to \infty$.

Proof. The validity of this Corollary is as a direct implication of Theorem 3 for $\alpha_n = \alpha, \beta_n = \beta, n \geq 1$.

■

Lemma 1 ([5]). Let $(L,(\cdot,\cdot|\cdot))$ be a 2-pre-Hilbert space.

a) If $x_n \to x$, $n \to \infty$, then $x_n \xrightarrow{w} x$, $n \to \infty$.

б) If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} x'$, $n \to \infty$, then $x = x'$. ■

Theorem 4. Let $(L,(\cdot,\cdot|\cdot))$ be a real 2-pre-Hilbert space. If $x_n \to x$, $y_n \to y$, $\alpha_n \to \alpha$ and $\beta_n \to \beta$, for $n \to \infty$, then

$$\lim_{n\to\infty}(\alpha_n x_n, \beta_n y_n | z) = (\alpha x, \beta y | z), \text{ for every } z \in L. \tag{2}$$

Proof. Let $x_n \to x$, $y_n \to y$, $\alpha_n \to \alpha$ and $\beta_n \to \beta$, for $n \to \infty$. Then, Theorem 1 a) implies

$$\lim_{n\to\infty} \| \alpha_n x_n - \alpha x, z \| = 0, \ \lim_{n\to\infty} \| \beta_n y_n - \beta y, z \| = 0, \text{ for every } z \in L. \tag{3}$$

Properties of 2-inner product and the Cauchy-Buniakowsky-Schwarz inequality imply

$$\begin{aligned} 0 &\leq |(\alpha_n x_n, \beta_n y_n | z) - (\alpha x, \beta y | z)|\\ &= |(\alpha_n x_n, \beta_n y_n | z) - (\alpha_n x_n, \beta y | z) + (\alpha_n x_n, \beta y | z) - (\alpha x, \beta y | z)|\\ &\leq |(\alpha_n x_n, \beta_n y_n | z) - (\alpha_n x_n, \beta y | z)| + |(\alpha_n x_n, \beta y | z) - (\alpha x, \beta y | z)|\\ &= |(\alpha_n x_n, \beta_n y_n - \beta y | z)| + |(\alpha_n x_n - \alpha x, \beta y | z)|\\ &\leq \| \alpha_n x_n, z \| \cdot \| \beta_n y_n - \beta y, z \| + \| \alpha_n x_n - \alpha x, z \| \cdot \| \beta y, z \| . \end{aligned}$$

Finally, letting $n \to \infty$, in the last inequality, by Theorem 2 b) and equality (3) follows the equality (2). ■

Corollary 2 ([5]). Let $(L,(\cdot,\cdot|\cdot))$ be a real 2-pre-Hilbert space. If $x_n \to x$ and $y_n \to y$, for $n \to \infty$, then $\lim_{n\to\infty}(x_n, y_n | z) = (x, y | z)$, for every $z \in L$.

Proof. The validity of this Corollary is as a direct implication of Theorem 4, $\alpha_n = \beta_n = 1, n \geq 1$. ■

Theorem 5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in 2-pre-Hilbert space L. If

$$\lim_{n\to\infty} \| x_n, y \| = \| x, y \| \text{ and } \lim_{n\to\infty}(x_n, x | y) = (x, x | y) \text{, for every } y \in L,$$

then $\lim_{n\to\infty} x_n = x$.

Proof. Let $y \in L$. Then

$$\begin{aligned} \| x_n - x, y \|^2 &= (x_n - x, x_n - x | y) \\ &= (x_n, x_n | y) - 2(x_n, x | y) + (x, x | y) \\ &= \| x_n, y \|^2 - 2(x_n, x | y) + \| x, y \|^2, \end{aligned}$$

So,

$$\begin{aligned} \lim_{n\to\infty} \| x_n - x, y \|^2 &= \lim_{n\to\infty} \| x_n, y \|^2 - 2 \lim_{n\to\infty}(x_n, x | y) + \| x, y \|^2 \\ &= 2 \| x, y \|^2 - 2(x, x | y) = 0. \end{aligned}$$

Finally, arbitrariness of y implies $x_n \to x$, $n \to \infty$. ■

A. White in [4] gave the term Cauchy sequence in a vector 2-normed space L of linearly independent vectors $y, z \in L$, i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ in a vector 2-normed space L is Cauchy sequence, if there exist $y, z \in L$ such that y and z are linearly independent and

$$\lim_{m,n\to\infty} \| x_n - x_m, y \| = 0 \text{ and } \lim_{m,n\to\infty} \| x_n - x_m, z \| = 0. \quad (4)$$

Further, White defines 2-Banach space, as a vector 2-normed space in which every Cauchy sequence is convergent and have proved that each 2-normed space with dimension 2 is 2-Banach space, if it is defined over complete field. The main point in this state is linearly independent of the vectors y and z, and by equalities (4) are sufficient for getting the proof. Later, this definition about Cauchy sequence into 2-normed space is reviewed, and the new definition is not contradictory with the state that 2-normed space with dimension 2 is 2-Banach space.

Definition 2 ([6]). The sequence $\{x_n\}_{n=1}^{\infty}$ in a vector 2-normed space L is *Cauchy sequence* if

$$\lim_{m,n\to\infty} \| x_n - x_m, y \| = 0 \text{, for every } y \in L. \quad (4)$$

Theorem 6. Let L be a vector 2-normed space.

i) If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in L, then for every $y \in L$ the real sequence $\{\| x_n, y \|\}_{n=1}^{\infty}$ is convergent.

ii) If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in L and $\{\alpha_n\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathbf{R}$, then $\{x_n + y_n\}_{n=1}^{\infty}$ and $\{\alpha_n x_n\}_{n=1}^{\infty}$ are Cauchy sequences in L.

Proof. The proof is analogous to the proof of Theorem 1.1, [4]. ■

Theorem 7. Let L be a vector 2-normed space, $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L and $\{x_{m_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. If $\lim_{k\to\infty} x_{m_k} = x$, then $\lim_{n\to\infty} x_n = x$.

Proof. Let $y \in L$ and $\varepsilon > 0$ is given. The sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and thus exist $n_1 \in \mathbf{N}$ such that $\| x_n - x_p, y \| < \frac{\varepsilon}{2}$, for $n, p > n_1$. Further, $\lim_{k\to\infty} x_{m_k} = x$, so exist $n_2 \in \mathbf{N}$ such that $\| x_{m_k} - x, y \| < \frac{\varepsilon}{2}$, for $m_k > n_2$. Letting $n_0 = \max\{n_1, n_2\}$ we get that for $n, m_k > n_0$ is true that

$$\| x_n - x, y \| \leq \| x_n - x_{m_k}, y \| + \| x_{m_k} - x, y \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

The arbitrariness of $y \in L$ implies $\lim_{n\to\infty} x_n = x$. ■

Theorem 8. Let L be a 2-pre-Hilbert space. If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in L, then for every $z \in L$ the sequence $\{(x_n, y_n \mid z)\}_{n=1}^{\infty}$ is Cauchy in $\mathbf{R}$.

Proof. Let $z \in L$. The properties of inner product and Cauchy-Buniakovski-Swartz inequality imply

$$\begin{aligned} |(x_n, y_n \mid z) - (x_m, y_m \mid z)| &\le |(x_n, y_n \mid z) - (x_m, y_n \mid z)| + |(x_m, y_n \mid z) - (x_m, y_m \mid z)| \\ &= |(x_n - x_m, y_n \mid z)| + |(x_m, y_n - y_m \mid z)| \\ &\le \| x_n - x_m, z \| \cdot \| y_n, z \| + \| x_m, z \| \cdot \| y_n - y_m, z \| . \end{aligned}$$

Finally, by the last inequality, equality (4) and Theorem 6 *i)* is true that for given $z \in L$ the sequence $\{(x_n, y_n \mid z)\}_{n=1}^{\infty}$ is Cauchy sequence. ■

III. CHARACTERIZATION OF 2-BANACH SPACES

Theorem 9. Each 2-normed space L with finite dimension is 2-Banach space.

Proof. Let L be a 2-normed space with finite dimension, $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L and $a \in L$. By $\dim L > 1$, exist $b \in L$ such that the set $\{a, b\}$ is linearly independent. By Theorem 1, [7],

$$\| x \|_{a,b,2} = (\| x, a \|^2 + \| x, b \|^2)^{1/2}, \; x \in L \tag{5}$$

defines norm in L, and by Theorem 6, [7] the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy into the space $(L, \| \cdot \|_{a,b,2})$. But normed space L with finite dimension is Banach, so exist $x \in L$ such that $\lim_{n\to\infty} \| x_n - x \|_{a,b,2} = 0$. So, by (5) we get

$$\lim_{n\to\infty} \| x_n - x, a \| = \lim_{n\to\infty} \| x_n - x, b \| = 0 . \tag{6}$$

We'll prove that for every $y \in L$,

$$\lim_{n\to\infty} \| x_n - x, y \| = 0 . \tag{7}$$

Clearly, if $y = \alpha a$, then (6) and the properties of 2-norm imply (7). Let the set $\{y, a\}$ be linearly independent. Then $(L, \| \cdot \|_{y,a,2})$ is a Banach space, and moreover because of the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in $(L, \| \cdot \|_{y,a,2})$, exist $x' \in L$ such that

$$\lim_{n\to\infty} \| x_n - x' \|_{y,a,2} = 0 . \tag{8}$$

But in space with finite dimension each two norms are equivalent (Theorem 2, [8], pp. 29), and thus exist $m, M > 0$ such that

$$m \| x_n - x' \|_{y,a,2} \le \| x_n - x' \|_{a,b,2} \le M \| x_n - x' \|_{y,a,2} ,$$

This implies $\lim_{n\to\infty} \| x_n - x' \|_{a,b,2} = 0$ and moreover $\lim_{n\to\infty} \| x_n - x \|_{a,b,2} = 0$. So, we may conclude $x' = x$. By (8), $\lim_{n\to\infty} \| x_n - x \|_{y,a,2} = 0$, and thus (7) holds. Finally the arbitrariness of y implies the validity of the theorem i.e. the proof of the theorem is completed. ■

Definition 3. Let L be a 2-normed space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in L. The series $\sum_{n=1}^{\infty} x_n$ is called as *summable* in L if the sequence of its partial sums $\{s_m\}_{m=1}^{\infty}$, $s_m = \sum_{n=1}^{m} x_n$ converges in L. If $\{s_m\}_{m=1}^{\infty}$ converges to s, then s is called as a sum of the series $\sum_{n=1}^{\infty} x_n$, and is noted as $s = \sum_{n=1}^{\infty} x_n$

The series $\sum_{n=1}^{\infty} x_n$ is called as *absolutely summable* in L if for each $y \in L$ holds $\sum_{n=1}^{\infty} \| x_n, y \| < \infty$.

Theorem 10. The 2-normed space L is 2-Banach if and only if each absolutely summable series of elements in L is summable in L.

Proof. Let L be a 2-Banach space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in L and for each $y \in L$ holds $\sum_{n=1}^{\infty} \| x_n, y \| < \infty$. Then the sequence $\{s_m\}_{m=1}^{\infty}$ of partial sums of the series $\sum_{n=1}^{\infty} x_n$ is Cauchy in L, because of

$$\| s_{m+k} - s_m, y \| = \| x_{m+1} + x_{m+2} + ... + x_{m+k}, y \| \leq \| x_{m+1}, y \| + \| x_{m+2}, y \| + ... + \| x_{m+k}, y \|,$$

for every $y \in L$ and for every $m, k \geq 1$. But, L is a 2-Banach space, and thus the sequence $\{s_m\}_{m=1}^{\infty}$ converge in L. That means, the series $\sum_{n=1}^{\infty} x_n$ is summable in L.

Conversely, let each absolutely summable series be summable in L. Let $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in L and $y \in L$. Let $m_1 \in \mathbf{N}$ is such that $\| z_m - z_{m_1}, y \| < 1$, for $m \geq m_1$. By induction we determine $m_2, m_3, ...$ such that $m_k < m_{k+1}$ and for each $m \geq m_k$ holds $\| z_m - z_{m_k}, y \| \leq \frac{1}{k^2}$. Letting $x_k = z_{m_{k+1}} - z_{m_k}$, $k = 1, 2, ...,$ we get $\sum_{k=1}^{\infty} \| x_k, y \| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. The arbitrarily of $y \in L$ implies the series $\sum_{k=1}^{\infty} x_k$ is absolutely summable. Thus, by assumption, the series $\sum_{k=1}^{\infty} x_k$ is summable in L. But, $z_{m_k} = z_{m_1} + \sum_{i=1}^{k-1} x_i$. Hence, the subsequence $\{z_{m_k}\}_{k=1}^{\infty}$ of the Cauchy sequence $\{z_n\}_{n=1}^{\infty}$ converge in L. Finally, by Theorem 7, the sequence $\{z_n\}_{n=1}^{\infty}$ converge in L. It means L is 2-Banach space. ■

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