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Analysis of a Batch Arrival and Batch Service Queuing System with Multiple Vacations

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ABSTRACT: *This paper concerns the queuing system $M^X/G^{1,\,B}/1$, to which the customers are assumed to arrive in batches of random size X according to a compound Poisson process and also are served in batches. As soon as the system becomes empty, the server leaves for a vacation of random length V. If no customers are available for service after returning from that vacation, the server keeps on taking vacations till he finds at least one customer in the queue, then immediately begins to serve the customers up to the service capacity B. If more than B customers are present when the server returns from a vacation, the first B customers are taken into service. If fewer than B customers are present, all waiting customers go into service. Late arrivals are not allowed to join the ongoing service. The steady state behavior of this queuing system is derived by an analytic approach to study the queue size distribution at a random point as well as a departure point of time under multiple vacation policy. It may be noted that the results in [5] and [7] can be obtained as special cases from the results in this paper.*

Keywords: *Queuing system, batch arrival, multiple vacations, batch service, queuing length*

I. Introduction

Most queuing models assume that customers are arrived singly and are served singly. But this assumption is far from truth when we consider those numerous real-world situations in which customers are arrived and are served in batches. We call such queues "batch arrival and batch service queues" (or bulk queues). Such queues frequently occur, for example, in loading and unloading of cargoes at a sea-port,

in traffic signal systems, in a mass transportation system, in digital communication, in computer network and in production/ inventory systems (e.g., Takagi [1991] and Doshi [1986, 1990a]).

Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. The major general result for vacation models is the stochastic decomposition result, which allows the system to be analyzed by considering separately the distribution of the system size with no vacations and the additional queue size due to vacations. This important result was first established by Fuhrmann and Cooper [1985] for generalized vacation as well as multiple vacation models, where the server keeps on taking a sequence of vacations of random length till it finds at least one unit in the system to start each busy period for the M/G/1 queuing systems. Later Doshi [1986] extended this result for the single vacation model through a different approach, where the server takes exactly one vacation at the end of each busy period. In this model if the server finds no units after returning from a vacation, he stays in the system waiting for a unit to arrive. Shanthikumar [1988] showed that the queue size decomposition holds even for the M/G/1 models with bulk arrival, reneging, balking etc. In terms of unfinished work in the system, Boxma and Groenendijk [1987] proved the decomposition result for the M/G/1 type vacation models. Recently Doshi [1990b] and Leung [1992] extended the results of Boxma and Groenendijk [1987], Harris and Marchal [1988] and Shathikumar [1988], proved the stochastic decomposition result for unfinished work in the system and additional delay due to the vacation times respectively in more general setting.

At present, however, most studies are devoted to batch arrival queues with vacation because of its interdisciplinary character. Considerable efforts have devoted to study these models by Baba [1986], Lee and Srinivasan [1989], Lee et.all [1994,1995], Borthakur and Choudhury [1997] of $M^X/G/1$ type queuing models of this nature have been served by Chae and Lee [1995] and Medhi [1997].

In this paper, we first study the steady state behavior of the queue size distribution for the $M^X/G^{1,B}/1$ queue with multiple vacation policy at the stationary point of time as well as departure point of time through an analytical approach. Also, we show that the departure point queue size distribution of this model can be expressed as convolution of the distributions of three independent random variables, one of which is the queue size of the standard $M^X/G^{1,B}/1$ queue without vacations. Efforts have also been made to interpret other two random variables properly.

II. $M^x/G^{1,B}/1$ Queuing System With Multiple Vacation

In this chapter we consider a bulk queuing system with following specifications:

1. As soon as a service is finished, the server takes into service a maximum of B customers, (B is called the service capacity).
2. If fewer than B customers are present at a service completion epoch, all the existing customers are taken into service.
3. Customers who arrive while a group of customers are being served cannot join the service even if there is space available.
4. The server takes a vacation of random length V if there are nosremotsuc to serve.
5. If no customers are available for service after the server returns from a vacation, he keeps on taking vacations till he finds at least one customer in the queue and begins to serve.
6. If the server finds at least one customer waiting for service when he returns from the vacation, the server immediately begins to serve the customers up to the capacity B.

We first write the system state equations for its stationary (random) queue size (including those in service, if any) distribution by treating elapsed service time and vacation time as supplementary variables. We now define the following notations and probabilities:

λ = arrival rate of batch ,

μ = service rate ,

B = service capacity ,

ρ = traffic intensity ,

X = group size random variable of arrival ,

c_k = p(x=k) ; $k \geq 1$,

$X(Z)$ = PGF of X ,

S = service time random variable ,

V = vacation time random variable ,

$S(x), V(x)$ = the probability distribution functions of "S", "V" ,

$s(x), v(X)$ = the probability density functions of "S", "V" ,

$S^*(s), V^*(s)$ = laplace stieltjes transforms of "$S(x)$", "$V(x)$" ,

S^0 = elapsed service time for the customer in service ,

V^0 = elapsed vacation time for the server on a vacation ,

$N_Q(t)$= queue size at time t .

Further, it is assumed that $V(0) = 0, V(\infty) = 1, S(0) = 0$ *and* $S(\infty) = 1$ and that $V(x)$ and $S(x)$ are continuous at x=0, so that

$$v(x)\,dx = dV(x)/[1 - V(x)]$$

and

$$s(x)\,dx = dS(x)/[1 - S(x)]$$

are the first order differential functions of "V" and "S" respectively.

Let $N_Q(t)$ be the queue size at time "t" and $S^0(t)$ be the elapsed service time at time "t". Further, we assume that $V^0(t)$ is the elapsed vacation time at time "t". Let us now introduce the following random variable

$$Y(t) = \begin{cases} 0, & \textit{if the server is on vacation at time "t"} \\ 1, & \textit{if the server is busy at time "t"} \end{cases}$$

So that the supplementary variables $S^0(t)$ and $V^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), L(t)\}$, where $L(t) = V^0(t)$ if $Y(t) = 0$ and $L(t) = S^0(t)$ if $Y(t) = 1$.

We define

$$\mathrm{p}_{0,m}(x)\,dx = lim_{t\to\infty}\,prob[N_Q(t) = m, L(t) = V^0(t); x < V^0(t) \le x + dx]\;;\quad x > 0, m \ge 0\;,$$

$$\mathrm{p}_{1,n}(x)\,dx = lim_{t\to\infty}\,prob[N_Q(t) = n, L(t) = S^0(t); x < S^0(t) \le x + dx]\;;\quad x > 0, n \ge 1\,.$$

Now the analysis of this queuing process at the stationary point of time can be done by using forward kolmogorov equations, which under the steady state conditions can be written as:

$$\left(\frac{d}{dx}\right)\mathrm{p}_{0,0}(x) + [\lambda + v(x)]\mathrm{p}_{0,0}(x) = 0\;; x > 0\;, \tag{2.1}$$

$$\left(\frac{d}{dx}\right)\mathrm{p}_{0,n}(x) + [\lambda + v(x)]\mathrm{p}_{0,n}(x) = \lambda\sum_{k=1}^{n} c_k \mathrm{p}_{0,n-k}(x)\;;\quad x > 0, n \ge 1\;, \tag{2.2}$$

$$\left(\frac{d}{dx}\right)\mathrm{p}_{1,n}(x) + [\lambda + s(x)]\mathrm{p}_{1,n}(x) = \lambda\sum_{k=0}^{n-1} c_k \mathrm{p}_{1,n-k}(x)\;;\quad x > 0, n \ge 1\;, \tag{2.3}$$

$$\mathrm{p}_{0,0}(0) = \int_0^\infty v(x)\mathrm{p}_{0,0}(x)\,dx + \sum_{i=1}^{B}\int_0^\infty s(x)\mathrm{p}_{1,i}(x)\,dx\,. \tag{2.4}$$

Where

$$\mathrm{p}_{0,0} = \int_0^\infty \mathrm{p}_{0,0}(x)\,dx\,.$$

These equations are to be solved under the following boundary conditions, at x=0:

$$\mathrm{p}_{0,0}(0) = \lambda\mathrm{p}_{0,0}\;, \tag{2.5}$$

$$\mathrm{p}_{0,n}(0) = 0\quad, n \ge 1\;, \tag{2.6}$$

$$\mathrm{p}_{1,n}(0) = \int_0^\infty v(x)\mathrm{p}_{0,n}(x)\,dx + \int_0^\infty s(x)\mathrm{p}_{1,n+B}(x)\,dx, n \ge 1, \tag{2.7}$$

and the normalizing condition

$$\sum_{n=0}^{\infty}\int_0^\infty \mathrm{p}_{0,n}(x)\,dx + \sum_{n=1}^{\infty}\int_0^\infty \mathrm{p}_{1,n}(x)dx = 1\,. \tag{2.8}$$

Let us define the following PGFS

$$\mathrm{P}_0(x;\mathrm{z}) = \sum_{n=0}^{\infty}\mathrm{z}^n\mathrm{p}_{0,n}(x)\;;\quad |\mathrm{z}| \le 1,$$

$$\mathrm{P}_0(0;\mathrm{z}) = \sum_{n=0}^{\infty}\mathrm{z}^n\mathrm{p}_{0,n}(0)\;;\quad |\mathrm{z}| \le 1,$$

$$\mathrm{P}_1(x;\mathrm{z}) = \sum_{n=1}^{\infty}\mathrm{z}^n\mathrm{p}_{1,n}(x)\;;\quad |\mathrm{z}| \le 1,$$

$$\mathrm{P}_1(0;\mathrm{z}) = \sum_{n=1}^{\infty}\mathrm{z}^n\mathrm{p}_{1,n}(0)\;;\quad |\mathrm{z}| \le 1.$$

Now multiplying equations (2.1) by z^0 and (2.2) by z^n and summing over possible values of n yield.

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathrm{z}^n\mathrm{p}_{0,n}(x) = \sum_{n=0}^{\infty} -[\lambda + v(x)]\mathrm{z}^n\mathrm{p}_{0,n}(x) + \lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n} c_k\mathrm{z}^n\mathrm{p}_{0,n-k}(x)\;,$$

From PGFS, we get

$$\frac{d}{dx}\mathrm{P}_0(x,z) = -[\lambda + v(x)]\mathrm{P}_0(x,z) + \lambda \sum_{n=1}^{\infty}\sum_{k=1}^{n} z^n c_k \mathrm{p}_{0,n-k}(x) \quad ,$$

Now

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} z^n c_k \mathrm{p}_{0,n-k}(x) = z c_1 \mathrm{p}_{0,0}(x) + \sum_{k=1}^{2} z^2\, c_k \mathrm{p}_{0,2-k}(x) + \sum_{k=1}^{3} z^3\, c_k \mathrm{p}_{0,3-k}(x) + \cdots$$

$$= z c_1 \mathrm{p}_{0,0}(x) + z^2 c_1 \mathrm{p}_{0,1}(x) + z^2 c_2 \mathrm{p}_{0,0}(x) + z^3 c_1 \mathrm{p}_{0,2}(x) + z^3 c_2 \mathrm{p}_{0,1}(x) + z^3 c_3 \mathrm{p}_{0,0}(x) + \cdots$$

$$= z c_1\{\mathrm{p}_{0,0}(x) + z\mathrm{p}_{0,1}(x) + z^2 \mathrm{p}_{0,2}(x) + z^3\, \mathrm{p}_{0,3}(x) + \cdots\} + z^2 c_2\{\mathrm{p}_{0,0}(x) + z\mathrm{p}_{0,1}(x) + \cdots\} + \cdots$$

$$= \sum_{k=1}^{\infty} z^k c_k \sum_{n=0}^{\infty} z^n \mathrm{p}_{0,n}(x) = \sum_{k=1}^{\infty} z^k c_k \mathrm{P}_0(x,z).$$

Let

$$X(z) = \sum_{k=1}^{\infty} z^k c_k,$$

then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} z^n c_k \mathrm{p}_{0,n-k}(x) = X(z)\,\mathrm{P}_0(x,z),$$

then

$$\frac{d}{dx}\mathrm{P}_0(x,z) = -[\lambda + v(x)]\mathrm{P}_0(x,z) + \lambda X(z)\mathrm{P}_0(x,z)$$
$$= [\lambda X(z) - \lambda - v(x)]\,\mathrm{P}_0(x,z),$$

$$\frac{d\mathrm{P}_0(x,z)}{\mathrm{P}_0(x,z)} = [\lambda X(z) - \lambda - v(x)]\, dx.$$

Integrating with respect to x

$$ln\mathrm{P}_0(x,z) = \lambda X(z)x - \lambda x - \int_0^x v(t)\, dt + constant$$

$$\mathrm{P}_0(x,z) = A.\, e^{\lambda X(z)x}.\, e^{-\lambda x}.\, exp\{-\int_0^x v(t)\, dt\}.$$

To find A put x=0

$$\mathrm{P}_0(0,z) = A = \sum_{n=0}^{\infty} z^n \mathrm{p}_{0,n}(0)$$
$$= \mathrm{p}_{0,0}(0) + \sum_{n=1}^{\infty} z^n \mathrm{p}_{0,n}(0).$$

Then

$$A = \mathrm{p}_{0,0}(0) = \lambda \mathrm{p}_{0,0}.$$

Then we have

$$\mathrm{P}_0(x,z) = \lambda \mathrm{p}_{0,0} exp\{-\int_0^x v(t)\, dt\}.\, e^{-\lambda[1-X(z)]x} \quad ,x > 0, \tag{2.9}$$

we have that

$$v(x)\, dx = \frac{dV(x)}{[1-V(x)]}, \qquad s(x)\, dx = \frac{dS(x)}{[1-S(x)]}.$$

Then

$$v(x)[1-V(x)] = \frac{dV(x)}{dx},\ v(x) - v(x)V(x) = \frac{dV(x)}{dx},$$

$$\frac{dV(x)}{dx} + v(x)V(x) = v(x).$$

This is a linear differential equation of the first order and to solve it multiply both sides by $e^{\int_0^x v(t)\,dt}$

$$\frac{dV(x)}{dx} e^{\int_0^x v(t)\,dt} + v(x)e^{\int_0^x v(t)\,dt} V(x) = v(x)e^{\int_0^x v(t)\,dt}$$

$$\frac{d}{dx}\left[V(x)e^{\int_0^x v(t)\,dt}\right] = v(x)e^{\int_0^x v(t)\,dt} \quad ,$$

by integrating both sides, we get

$$V(x)e^{\int_0^x v(t)\,dt} = \int_0^x v(u)e^{\int_0^u v(t)\,dt}\,du$$

$$= \int_0^x de^{\int_0^u v(t)\,dt}$$

$$= e^{\int_0^x v(t)\,dt} - 1 \quad ,$$

Then

$$V(x) = 1 - e^{-\int_0^x v(t)\,dt} \quad .$$

Substituting this into equation (2.9), we have

$$\mathrm{P}_0(x,z) = \lambda \mathrm{p}_{0,0}[1-V(x)]e^{-\lambda[1-X(z)]x} \quad ,x > 0 \ .$$

Then

$$\mathrm{P}_0(z) = \int_0^\infty \mathrm{P}_0(x,z)\,dx$$

$$= \int_0^\infty \lambda \mathrm{p}_{0,0}[1-V(x)]e^{-\lambda[1-X(z)]x}\,dx$$

$$= \lambda \mathrm{p}_{0,0} \int_0^\infty [1-V(x)]e^{-\lambda[1-X(z)]x}\,dx \quad ,$$

integrating by parts, we get

$$\mathrm{P}_0(z) = \lambda \mathrm{p}_{0,0} \frac{1-V^*[\lambda-\lambda X(z)]}{\lambda[1-X(z)]}$$

$$\mathrm{P}_0(z) = \mathrm{p}_{0,0} \frac{1-V^*[\lambda-\lambda X(z)]}{[1-X(z)]} \quad .$$

Similarly, multiplying equation (2.3) by z^n and summing over all values of n we obtain

$$\frac{d}{dx}\mathrm{P}_1(x,z) = -[\lambda+s(x)]\,\mathrm{P}_1(x,z)$$

$$+\lambda \sum_{n=1}^\infty \sum_{k=1}^n c_k \mathrm{p}_{1,n-k}(x) z^n$$

$$= -[\lambda+s(x)]\,\mathrm{P}_1(x,z) + \lambda X(z)\,\mathrm{P}_1(x,z)$$

$$= [\lambda X(z) - \lambda - s(x)]\,\mathrm{P}_1(x,z)$$

$$\frac{d\,\mathrm{P}_1(x,z)}{\mathrm{P}_1(x,z)} = [\lambda x(z) - \lambda - s(x)]\,dx\,,$$

by integrating with respect to x

$$ln\mathrm{P}_1(x,z) = \lambda X(z)x - \lambda x - \int_0^x s(t)\,dt + constant$$

$$\mathrm{P}_1(x,z) = B.\,e^{\lambda X(z)x}.\,e^{-\lambda x}.\,exp\{-\int_0^x s(t)\,dt\}$$

$$= B.\,e^{-\lambda[1-X(z)]x} exp\{-\int_0^x s(t)\,dt\}$$

$$= B[1 - S(x)].\,e^{-\lambda[1-X(z)]x}$$

To find B put x=0

$$\mathrm{P}_1(0,z) = B\quad,$$

then

$$\mathrm{P}_1(x,z) = \mathrm{P}_1(0,z)[1 - S(x)].\,e^{-\lambda[1-X(z)]x}\;\;,\;\; x > 0\,. \tag{2.10}$$

Now

$$\mathrm{P}_1(0,z) = \sum_{n=1}^{\infty} z^n \mathrm{p}_{1,n}(0)$$

From equation (2.7)

$$= \sum_{n=1}^{\infty} z^n [\int_0^{\infty} v(x)\mathrm{p}_{0,n}(x)\,dx$$

$$+ \int_0^{\infty} s(x)\mathrm{p}_{1,n+B}(x)dx]$$

$$= \int_0^{\infty} [\sum_{n=1}^{\infty} z^n\,\mathrm{p}_{0,n}(x)]v(x)dx$$

$$+ \int_0^{\infty} [\sum_{n=1}^{\infty} z^n\,\mathrm{p}_{1,n+B}(x)]s(x)dx$$

$$= \int_0^{\infty} [\mathrm{P}_0(x,z) - \mathrm{p}_{0,0}(x)]v(x)dx$$

$$+ \frac{1}{z^B}\int_0^{\infty} [\sum_{n=1}^{\infty} z^{n+B}\,\mathrm{p}_{1,n+B}(x)]s(x)dx$$

$$= \int_0^{\infty} [\mathrm{P}_0(x,z) - \mathrm{p}_{0,0}(x)]v(x)dx$$

$$+ \frac{1}{z^B}\int_0^{\infty} [\mathrm{P}_1(x,z) - \sum_{i=1}^{B} z^i\,\mathrm{p}_{1,i}(x)]s(x)dx$$

$$= \int_0^{\infty} \mathrm{P}_0(x,z)v(x)dx - \int_0^{\infty} \mathrm{p}_{0,0}(x)v(x)dx$$

$$+ \frac{1}{z^B}\int_0^{\infty} \mathrm{P}_1(x,z)s(x)dx$$

$$- \frac{1}{z^B}\sum_{i=1}^{B} z^i \int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx$$

From equation (2.4)

$$= \int_0^{\infty} \mathrm{P}_0(x,z)v(x)dx$$

$$-[\lambda \mathrm{p}_{0,0} - \sum_{i=1}^{B} \int_0^{\infty} \mathrm{p}_{1,i}(x)s(x)dx]$$

$$+ \frac{1}{z^B}\int_0^{\infty} \mathrm{P}_1(x,z)s(x)dx$$

$$-\frac{1}{z^B}\sum_{i=1}^{B} z^i \int_0^\infty p_{1,i}(x)\, s(x) dx$$

$$= \int_0^\infty P_0(x,z)v(x)dx - \lambda p_{0,0}$$

$$+\frac{1}{z^B}\int_0^\infty P_1(x,z)s(x)dx$$

$$+\frac{1}{z^B}\sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)\, s(x)dx$$

From equations (2.9) and (2.10)

$$= \int_0^\infty \lambda p_{0,0}\,[1 - V(x)]e^{-\lambda[1-X(z)]x} v(x)\, dx$$

$$+\frac{1}{z^B}\int_0^\infty P_1(0,z)\,[1 - S(x)]e^{-\lambda[1-X(z)]x} s(x)\, dx$$

$$-\lambda p_{0,0}$$

$$+\frac{1}{z^B}\sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)\, s(x)dx$$

$$= \int_0^\infty \lambda p_{0,0}\, e^{-\lambda[1-X(z)]x} dV(x)$$

$$+\frac{1}{z^B}\int_0^\infty P_1(0,z)\, e^{-\lambda[1-X(z)]x} dS(x)$$

$$-\lambda p_{0,0}$$

$$+\frac{1}{z^B}\sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)\, s(x)dx$$

$$= \lambda p_{0,0}\, V^*[\lambda - \lambda X(z)]$$

$$+\frac{1}{z^B}P_1(0,z)S^*[\lambda - \lambda X(z)]$$

$$-\lambda p_{0,0} + \frac{1}{z^B}\sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)\, s(x)dx$$

Multiply by z^B to get

$$P_1(0,z)S^*[\lambda - \lambda X(z)] - z^B P_1(0,z)$$

$$= z^B\lambda p_{0,0} - z^B\lambda p_{0,0}V^*[\lambda - \lambda X(z)]$$

$$-\sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)\, s(x)dx$$

Then

$$P_1(0,z) = \frac{\lambda z^B p_{0,0}\{1 - V^*[\lambda - \lambda X(z)]\} - \sum_{i=1}^{B}(z^B - z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{S^*[\lambda - \lambda X(z)] - z^B}.$$

Therefore, we get

$$P_1(z) = \int_0^\infty P_1(x,z)\, dx$$

Substituting from equation (2.10) to obtain

$$P_1(z) = \int_0^\infty P_1(0,z)\,[1 - S(x)]e^{-\lambda[1-X(z)]x}\, dx$$

$$= \frac{\lambda z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}-\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{S^*[\lambda-\lambda X(z)]-z^B}$$

$$\int_0^\infty [1-S(x)]\, e^{-\lambda[1-X(z)]x}\, dx$$

Integrating by parts we get

$$P_1(z) = \left[\frac{\lambda z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}-\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{S^*[\lambda-\lambda X(z)]-z^B}\right]$$

$$\left[\frac{1-S^*[\lambda-\lambda X(z)]}{\lambda-\lambda X(z)}\right]$$

$$= \left[\frac{\lambda z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}}{S^*[\lambda-\lambda X(z)]-z^B}\right]\left[\frac{1-S^*[\lambda-\lambda X(z)]}{\lambda-\lambda X(z)}\right]$$

$$-\left[\frac{\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{S^*[\lambda-\lambda X(z)]-z^B}\right]\left[\frac{1-S^*[\lambda-\lambda X(z)]}{\lambda-\lambda X(z)}\right]$$

Then

$$P_1(z) = \frac{z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}\{1-S^*[\lambda-\lambda X(z)]\}}{\{1-X(z)\}\{S^*[\lambda-\lambda X(z)]-z^B\}}$$

$$-\frac{\{1-S^*[\lambda-\lambda X(z)]\}\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{\{\lambda-\lambda X(z)\}\{S^*[\lambda-\lambda X(z)]-z^B\}}\ .$$

Using the normalizing condition (2.8) and taking the limit of $[P_0(z)+P_1(z)]$ as $z \to 1$, we get

$P_0(1) = \frac{0}{0}$. By L'Hopital's rule, we get

$$P_0(z) = \frac{\lambda X^{\backslash}(z) V^{*\backslash}[\lambda-\lambda X(z)]}{-X^{\backslash}(z)} p_{0,0}$$

$at\ z = 1$

$$P_0(1) = -\lambda V^{*\backslash}(0) p_{0,0} = \lambda E(V) p_{0,0} \qquad \text{(i)}$$

Also,

$P_1(1) = \frac{0}{0}$. By L'Hopital's rule, we get

$P_1(z)|_{z=1} = \frac{0}{0}$ By applying L'Hopital's rule another time, we get

$at\ z = 1$

$$P_1(1) = \frac{2\lambda^2 p_{0,0} V^{*\backslash}(0) X^{\backslash 2}(1) S^{*\backslash}(0)}{2X^{\backslash}(1)\{\lambda X^{\backslash}(1) S^{*\backslash}(0)+B\}}$$

$$-\frac{2\lambda X^{\backslash}(1) S^{*\backslash}(0)\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)s(x)dx}{2\lambda X^{\backslash}(1)\{\lambda X^{\backslash}(1) S^{*\backslash}(0)+B\}}$$

$$= \frac{\lambda^2 p_{0,0} E(V) E(X) E(S)}{\{-\lambda E(X) E(S)+B\}}$$

$$+\frac{E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{\{-\lambda E(X)E(S)+B\}}$$

then

$$\mathbf{P}_1(1)=\frac{\lambda^2\mathbf{p}_{0,0}E(V)E(X)E(S)+E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{B-\lambda E(X)E(S)} \quad \text{(ii)}$$

Now using the normalizing condition to obtain $\mathbf{p}_{0,0}$ as follows:

$$\mathbf{P}_0(1)+\mathbf{P}_1(1)=1$$

$$\lambda E(V)\mathbf{p}_{0,0}+\frac{\lambda^2\mathbf{p}_{0,0}E(V)E(X)E(S)+E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{B-\lambda E(X)E(S)}=1$$

$$\lambda E(V)\mathbf{p}_{0,0}[B-\lambda E(X)E(S)]+\lambda^2\mathbf{p}_{0,0}E(V)E(X)E(S)+E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)\,s(x)dx=B-\lambda E(X)E(S)$$

$$\lambda BE(V)\mathbf{p}_{0,0}+E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)\,s(x)dx=B-\lambda E(X)E(S)$$

Then

$$\mathbf{p}_{0,0}=\frac{B-\lambda E(X)E(S)-E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{\lambda BE(V)}.$$

Let

$$\rho=\frac{\lambda E(X)E(S)}{B},$$

Then

$$\mathbf{p}_{0,0}=\frac{B(1-\rho)-E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{\lambda BE(V)}.$$

Let $\mathbf{P}_0(z)+\mathbf{P}_1(z)=\mathbf{P}(z)$ be the PGF of the stationary queue size distribution of this $M^X/G^{1,B}/1$ multiple vacation model, then

$$\mathbf{P}(z)=\mathbf{p}_{0,0}\frac{1-V^*[\lambda-\lambda X(z)]}{[1-X(z)]}+\frac{z^B\mathbf{p}_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}\{1-S^*[\lambda-\lambda X(z)]\}}{\{1-X(z)\}\{S^*[\lambda-\lambda X(z)]-z^B\}}-\frac{\{1-S^*[\lambda-\lambda X(z)]\}\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{\{\lambda-\lambda X(z)\}\{S^*[\lambda-\lambda X(z)]-z^B\}}$$

$$=\frac{\left\{B(1-\rho)-E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx\right\}\{1-V^*[\lambda-\lambda X(z)]\}}{\lambda BE(V)[1-X(z)]}\left[1+\frac{z^B\{1-S^*[\lambda-\lambda X(z)]\}}{\{S^*[\lambda-\lambda X(z)]-z^B\}}\right]-\frac{\{1-S^*[\lambda-\lambda X(z)]\}\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx}{\{\lambda-\lambda X(z)\}\{S^*[\lambda-\lambda X(z)]-z^B\}}$$

Then

$$\mathbf{P}(z)=\frac{\left\{B(1-\rho)-E(S)\sum_{i=1}^{B}(B-i)\int_0^\infty \mathbf{p}_{1,i}(x)s(x)dx\right\}(1-z^B)S^*[\lambda-\lambda X(z)]}{S^*[\lambda-\lambda X(z)]-z^B}\left[\frac{\{1-V^*[\lambda-\lambda X(z)]\}}{BE(V)[\lambda-\lambda X(z)]}\right]$$

$$-[\frac{\{1-S^*[\lambda-\lambda X(z)]\}}{\lambda-\lambda X(z)}][\frac{\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)dx}{S^*[\lambda-\lambda X(z)]-z^B}] . \quad (2.11)$$

SPECIAL CASE:(I)

When B=1, then

$$P(z) = \frac{(1-\rho)(1-z)S^*[\lambda-\lambda X(z)]\{1-V^*[\lambda-\lambda X(z)]\}}{\{S^*[\lambda-\lambda X(z)]-z\}E(V)[\lambda-\lambda X(z)]} ,$$

$$P(z) = [\frac{(1-\rho)(1-z)S^*[\lambda-\lambda X(z)]}{\{S^*[\lambda-\lambda X(z)]-z\}}][\frac{1-V^*[\lambda-\lambda X(z)]}{E(V)[\lambda-\lambda X(z)]}]$$

$$= P(M^X/G/1\,;z)\varsigma(z) .$$

Where

$P(M^X/G/1\,;z) =$ The PGF of the stationary queue size distribution of the standard $M^X/G/1$ queue without vacation. This is well known Pollaczek-Khinchine formula for $M^X/G/1$ queue.

$$= [\frac{(1-\rho)(1-z)S^*[\lambda-\lambda X(z)]}{\{S^*[\lambda-\lambda X(z)]-z\}}],$$

and $\varsigma(Z) =$ The PGF of the number of units that arrive during the residual life of the vacation time

$$= [\frac{1-V^*[\lambda-\lambda X(z)]}{E(V)[\lambda-\lambda X(z)]}] .$$

Differentiating equation (2.2.11) w.r.t. Z and taking the limit as $Z \to 1$, we get

$$L_S = \left.\frac{dP(z)}{d(z)}\right|_{z=1}$$

By applying L'Hopital's rule four of times, we get

$$L_S = \frac{\lambda^2 p_{0,0}E(X)E(V^2)}{2}$$
$$+p_{0,0}\{\frac{\lambda E(V)[\lambda^2E^2(X)E(S^2)+\lambda E(S)E(X^2-X)-\rho B(B-1)]}{2B(1-\rho)^2} +$$
$$\frac{2\lambda B\rho E(V)+\lambda^2\rho E(X)E(V^2)}{2(1-\rho)}\}$$

$$+\frac{[\lambda BE(X)E(S^2)+\lambda E^2(S)E(X^2-X)-B(B-1)E(S)]\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)s(x)dx}{2B^2(1-\rho)^2}$$
$$+\frac{E(S)\sum_{i=1}^{B}(B(B-1)-i(i-1))\int_0^\infty p_{1,i}(x)s(x)dx}{2B(1-\rho)} . \quad (2.12)$$

SPECIAL CASE: (II)

Substituting with value $p_{0,0}$ into equation (2.12), then put B=1, we get

$$L_S = \lambda E(X)E(V_R) + \rho + \frac{\lambda^2E^2(X)E(S^2)+\lambda E(S)E(X^2-X)}{2(1-\rho)} ;$$

Where $E(V_R) = E(V^2)/2E(V)$ is the mean residual vacation time and L_S is the mean of the stationary queue size distribution of this model.

III. Departure Point Queue Size Distribution

In this section we derive the PGF of the departure point queue size distribution of the $M^X/G^{1,B}/1$ queue with multiple vacations. Following the argument of PASTA [see Wolff (1982)] we note that a departing customer will see 'j' units in the system just after a departure if and only if there were (j+B) units in the queue just before the departure and therefore we may write

$$P^+_j = A_0 \int_0^\infty s(x) p_{1,j+B}(x)\, dx; \qquad j \geq 0,$$

Where A_0 is a normalizing constant and $P^+_j = prob$[A departing customer will see "j" units in the system just after the departure]; j≥0.

Let $P^+(z)$ be the PGF of $\{P^+_j;\ j=0,1,2,\dots\}$, then

$$P^+(z) = A_0 \sum_{j=0}^\infty z^j \int_0^\infty s(x) p_{1,j+B}(x)\, dx$$

$$= A_0 \frac{1}{z^B} \int_0^\infty \sum_{j=0}^\infty \{z^{j+B} p_{1,j+B}(x)\} s(x)\, dx$$

$$= A_0 \frac{1}{z^B} \int_0^\infty \{P_1(x,z) - \sum_{i=1}^{B-1} z^i p_{1,i}(x)\} s(x)\, dx$$

$$= A_0 \frac{1}{z^B} \int_0^\infty s(x) P_1(x,z)\, dx$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx \quad .$$

From equation (2.10), we get

$$= A_0 \frac{1}{z^B} \int_0^\infty s(x)\, P_1(0,z)[1 - S(x)].\, e^{-\lambda[1-X(z)]x}\, dx$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx$$

$$= A_0 \frac{1}{z^B} \int_0^\infty P_1(0,z)[1 - S(x)].\, e^{-\lambda[1-X(z)]x} \frac{dS(x)}{1-S(x)}$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx$$

$$= A_0 \frac{1}{z^B} P_1(0,z) S^*[\lambda - \lambda X(z)]$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx$$

$$= A_0 \frac{1}{z^B} S^*[\lambda - \lambda X(z)]$$

$$\left[\frac{\lambda z^B p_{0,0}\{1 - V^*[\lambda - \lambda X(z)]\} - \sum_{i=1}^{B} (z^B - z^i) \int_0^\infty p_{1,i}(x) s(x) dx}{S^*[\lambda - \lambda X(z)] - z^B}\right]$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx$$

$$= A_0 \frac{1}{z^B} \frac{\lambda z^B p_{0,0}\{1 - V^*[\lambda - \lambda X(z)]\} S^*[\lambda - \lambda X(z)]}{S^*[\lambda - \lambda X(z)] - z^B}$$

$$- A_0 \frac{1}{z^B} \frac{S^*[\lambda - \lambda X(z)] \sum_{i=1}^{B} (z^B - z^i) \int_0^\infty p_{1,i}(x) s(x) dx}{S^*[\lambda - \lambda X(z)] - z^B}$$

$$- A_0 \frac{1}{z^B} \sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx$$

$$= A_0 \frac{1}{z^B} \frac{\lambda z^B p_{0,0}\{1 - V^*[\lambda - \lambda X(z)]\} S^*[\lambda - \lambda X(z)]}{S^*[\lambda - \lambda X(z)] - z^B}$$

$$- A_0 \frac{1}{z^B} \Big[\frac{S^*[\lambda - \lambda X(z)] \sum_{i=1}^{B} (z^B - z^i) \int_0^\infty p_{1,i}(x) s(x) dx}{S^*[\lambda - \lambda X(z)] - z^B} +$$

$$\sum_{i=1}^{B-1} z^i \int_0^\infty s(x) p_{1,i}(x)\, dx\Big] \quad .$$

Then

$$P^+(z) = \frac{A_0 S^*[\lambda-\lambda X(z)]\{\lambda z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\}-\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)s(x)}{z^B\{S^*[\lambda-\lambda X(z)]-z^B\}} - \frac{A_0\sum_{i=1}^{B-1} z^i \int_0^\infty s(x)p_{1,i}(x)\,dx]}{z^B} \quad . \qquad (3.1)$$

Now from $\lim_{z\to1} P^+(z) = 1$, we have

$$\lim_{z\to1} P^+(z) = \frac{0}{0} - A_0 \sum_{i=1}^{B-1}\int_0^\infty s(x)p_{1,i}(x)\,dx$$

Use L'Hopital's rule we obtain

$P^+(z)$

$$= \frac{\begin{array}{r} A_0 S^{*\backslash}[\lambda-\lambda X(z)][-\lambda X^{\backslash}(z)][\lambda z^B p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\} \\ -\sum_{i=1}^{B}(z^B-z^i)\int_0^\infty p_{1,i}(x)\,s(x)dx] \\ +A_0 S^*[\lambda-\lambda X(z)][\lambda B z^{B-1}p_{0,0}\{1-V^*[\lambda-\lambda X(z)]\} \\ +\lambda z^B p_{0,0}\{-V^{*\backslash}[\lambda-\lambda X(z)][-\lambda X^{\backslash}(z)]\} \\ -\sum_{i=1}^{B}(Bz^{B-1}-iz^{i-1})\int_0^\infty p_{1,i}(x)\,s(x)dx] \end{array}}{Bz^{B-1}\{S^*[\lambda-\lambda X(z)]-z^B\}+z^B\{S^{*\backslash}[\lambda-\lambda X(z)][-\lambda X^{\backslash}(z)]-Bz^{B-1}\}}$$

$$- \frac{A_0\sum_{i=1}^{B-1} z^i\int_0^\infty s(x)p_{1,i}(x)\,dx]}{z^B}$$

at $z = 1$

$$\lim_{z\to1} P^+(z) = P^+(1) = 1$$

$$= \frac{A_0[-\lambda^2 p_{0,0}E(V)E(X)-\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)s(x)dx}{\lambda E(S)E(X)-B}$$

$$-A_0\sum_{i=1}^{B-1}\int_0^\infty s(x)p_{1,i}(x)\,dx$$

$$A_0\{-\lambda^2 p_{0,0}E(V)E(X)-\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)\,s(x)dx$$
$$-[\lambda E(S)E(X)-B]\sum_{i=1}^{B-1}\int_0^\infty s(x)p_{1,i}(x)\,dx\,\}$$
$$= \lambda E(S)E(X) - B$$

$$A_0 = \frac{\lambda E(S)E(X)-B}{\begin{array}{c}-\{\lambda^2 p_{0,0}E(V)E(X)+\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)s(x)dx \\ +[\lambda E(S)E(X)-B]\sum_{i=1}^{B-1}\int_0^\infty s(x)p_{1,i}(x)\,dx\,\}\end{array}}$$

Since $\rho = \dfrac{\lambda E(S)E(X)}{B}$

$$A_0 = \frac{-(B\rho-B)}{\begin{array}{c}[\lambda^2 p_{0,0}E(V)E(X)+\sum_{i=1}^{B}(B-i)\int_0^\infty p_{1,i}(x)s(x)dx] \\ +[B\rho-B]\sum_{i=1}^{B-1}\int_0^\infty s(x)p_{1,i}(x)\,dx\end{array}}$$

$$\mathbf{A_0} = \frac{1}{\frac{\lambda^2 \mathrm{p}_{0,0} E(V)E(X) + \sum_{i=1}^{B}(B-i)\int_0^\infty \mathrm{p}_{1,i}(x)s(x)dx}{B(1-\rho)} - \sum_{i=1}^{B-1}\int_0^\infty \mathrm{s}(x)\mathrm{p}_{1,i}(x)\,\mathrm{d}x} \quad . \quad (3.2)$$

Hence from equations (3.1) and (3.2), we have

$$P^+(z) = \frac{1}{\frac{\lambda^2 \mathrm{p}_{0,0} E(V)E(X) + \sum_{i=1}^{B}(B-i)\int_0^\infty \mathrm{p}_{1,i}(x)s(x)dx}{B(1-\rho)} - \sum_{i=1}^{B-1}\int_0^\infty \mathrm{s}(x)\mathrm{p}_{1,i}(x)\,\mathrm{d}x}$$

$$\left[\frac{S^*[\lambda-\lambda X(z)][\lambda z^B \mathrm{p}_{0,0}\{1-V^*[\lambda-\lambda X(z)]\} - \sum_{i=1}^{B}(z^B - z^i)\int_0^\infty \mathrm{s}(x)\mathrm{p}_{1,i}(x)\,\mathrm{d}x\,]}{z^B\{S^*[\lambda-\lambda X(z)]-z^B\}} - \frac{\sum_{i=1}^{B-1} z^i \int_0^\infty \mathrm{s}(x)\mathrm{p}_{1,i}(x)\,\mathrm{d}x}{z^B}\right] \quad (3.3)$$

SPECIAL CASE:(III)
when B=1 then

$$P^+(z) = \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}S^*[\lambda-\lambda X(z)]}{\lambda E(V)E(X)\,[S^*[\lambda-\lambda X(z)]-z]}, \quad (3.4)$$

and in particular, if we let the service time distribution is exponential, i.e., if we take $S(x) = 1 - e^{-\mu x}$ $, x > 0,$ then $E(S) = \frac{1}{\mu}$ and $S^*(s) = \frac{\mu}{\mu+s}$ and therefore equation (3.4) simply reduces to

$$P^+(z) = \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}\,\frac{\mu}{\mu+\lambda(1-X(z))}}{\lambda E(V)E(X)\,[\frac{\mu}{\mu+\lambda(1-X(z))}-z]}$$

$$= \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}}{\lambda E(V)E(X)\,[1-z\,\frac{\mu+\lambda(1-X(z))}{\mu}]}$$

$$= \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}}{\lambda E(V)E(X)\,[1-z-\frac{\lambda}{\mu}z\,(1-X(z))]}$$

$$= \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}}{\lambda E(V)E(X) - \lambda z E(V)E(X) - \frac{\lambda^2}{\mu}E(V)E(X)\,z\,(1-X(z))]},$$

in this case $\rho = \frac{\lambda}{\mu}\ E(X)$, and so

$$P^+(z) = \frac{(1-\rho)\{1-V^*[\lambda-\lambda X(z)]\}}{\lambda E(V)[E(X)(1-z)-\rho\,z\,(1-X(z))]} \quad . \quad (3.5)$$

Note that this result was obtained earlier by Borthakur and Choudhury (1977) for the $M^X/M/1$ queuing system. One of the principle motivations for considering such a model is to utilize the idle time for secondary tasks such as maintenance work or machine breakdown. Atypical example has been reported by Levy and Yechiali (1975) in the form of multiple vacation model. Now, if we Take limit $\mu \to \infty$, for fixed λ and $E(X)$, then $\lim_{\mu\to\infty}\rho = 0$, and therefore from equation (2.3.5) we have

$$\lim_{\mu\to\infty} P^+(z) = \frac{1-V^*[\lambda-\lambda X(z)]}{\lambda E(X)E(V)(1-z)}$$

$$\lim_{\mu\to\infty} P^+(z) = \frac{1-\alpha(z)}{E(\alpha)(1-z)} \quad . \quad (3.6)$$

Where

$\alpha(z) = \sum_{j=0}^{\infty} z^j a_j$ is the PGF of the number of tagged customer that arrive in the system at vacation termination point of time .

$$= V^*[\lambda - \lambda X(z)]$$

and

$$E(\alpha) = \alpha^{\backslash}(1) = \lambda E(X)E(V) .$$

It may be noted here that the equation (3.6) represents the PGF of the additional queue size distribution due to idle of the multiple vacation model [e.g. see Borthakur and Choudhury (1997)] and this complies with the stochastic decomposition results of Fuhrmann and Cooper (1985) for the single unit arrival case. Thus the PGF of the $M^X/G/1$ queue with multiple vacation models can also be written as

$$P^+(z) = \frac{1-\alpha(z)}{E(\alpha)(1-z)}\ P(\mathrm{M^X/G/1}; z) . \qquad (3.7)$$

Now differentiating equation (2.3.3) w.r.t z and taking limit as $z \to 1$, we get

$$L_Q = \left.\frac{dP^+(z)}{d(z)}\right|_{z=1}$$

By applying L'Hopital's rule second of times, we get

$$\begin{aligned} L_Q = \mathrm{A_0}\{&2B^2\rho(1-\rho)\lambda^2 \mathrm{p}_{0,0} E(V)E(X) \\ &+ \textstyle\sum_{i=1}^{B}(B-i)\int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx \\ &+\lambda^3 B(1-\rho)\mathrm{p}_{0,0}E(V^2)E^2(X) \\ &+\lambda^2 B(1-\rho)\mathrm{p}_{0,0}E(V)E(X^2-X) \\ &+B(1-\rho)\textstyle\sum_{i=1}^{B}(B(B-1)-i(i-1)) \\ &\textstyle\int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx \\ &-2B^2(1-\rho)\textstyle\sum_{i=1}^{B}(B-i)\int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx \\ &+[\lambda^2E^2(X)E(S^2)+\lambda E(X^2-X)E(S)-B(B-1)] \\ &[\lambda^2\mathrm{p}_{0,0}E(V)E(X)+\textstyle\sum_{i=1}^{B}(B-i)\int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx] \\ &-2B^2(1-\rho)^2\textstyle\sum_{i=1}^{B-1}(i-B)\int_0^{\infty} \mathrm{p}_{1,i}(x)\,s(x)dx\} \\ &/2B^2(1-\rho)^2 . \end{aligned} \qquad (3.8)$$

SPECIAL CASE:(IV)

From equation (3.2) and substituting with value $\mathrm{p}_{0,0}$ into equation (3.8), then put B=1, we get

$$L_Q = \lambda E(X)E(V_R) + \rho + \frac{\lambda^2E^2(X)E(S^2)+\lambda E(S)E(X^2-X)}{2(1-\rho)} + E(X_R)$$

$$L_Q = L_S + E(X_R) ;$$

where L_Q is the mean queue size at departure point of time and $E(X_R) = 1/2\left[\frac{E(X^2)}{E(X)} - 1\right]$ is the mean residual batch size. This shows that $L_Q > L_S$ and equality holds iff $(X_R) = 0$.

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