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On ranges and null spaces of a special type of operator named $\lambda - jection$. – Part I

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ABSTRACT: *In this article, $\lambda - jection$ has been introduced which is a generalization of trijection operator as introduced in P.Chandra's Ph. D. thesis titled "Investigation into the theory of operators and linear spaces" (Patna University,1977). We obtain relation between ranges and null spaces of two given $\lambda - jections$ under suitable conditions.*

Key Words: *projection, trijection, $\lambda - jection$*

I. Introduction

Dr. P. Chandra has defined a trijection operator in his Ph.D. thesis titled " Investigation into the theory of operators and linear spaces". [1]. A projection operator E on a linear space X is defined as $E^2 = E$ as given in Dunford and Schwartz [2] , p .37 and Rudin [3], p.126. In analogue to this , E is a trijection operator if $E^3 = E$. It is a generalization of projection operator in the sense that every projection is a trijection but a trijection is not necessarily a projection.

II. Definition

Let X be a linear space and E be a linear operator on X. We call E a $\lambda - jection$ if
$E^3 + \lambda E^2 = (1+\lambda)E, \lambda$ being a scalar. Thus if $\lambda = 0$
, $E^3 = E \ i.e. E \ is \ a$ trijection. We see that $E^2 = E \Rightarrow E^3 = E$ and above condition is satisfied. Thus a projection is also a $\lambda - jection$.

III. Main Results

3.1 We first investigate the case when an expression of the form $aE^2 + bE$ is a projection where E is a $\lambda - jection$. For this we need

$(aE^2 + bE)^2 = aE^2 + bE$.

$\Rightarrow a^2E^4 + b^2E^2 + 2abE^3 = aE^2 + bE$ ……………………………………………………………(1)

$From\ definition\ of \lambda - jection$,

$E^3 = (1+\lambda)E - \lambda E^2$

$so\ E^4 = E.E^3 = (1+\lambda)E^2 - \lambda E^3$

$$= (1+\lambda)E^2 - \lambda\{(1+\lambda)E - \lambda E^2\}$$

$$= (1+\lambda+\lambda^2)E^2 - \lambda(1+\lambda)E$$

We put these values in (1) and after simplifying

$$\{a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda)\}E^2$$

$+\{(2ab - a^2\lambda)(1+\lambda) - b\}E = 0$

Equating Coefficients of E & E^2 to be 0, we get

$a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda) = 0$ ……………………………………………………………(2)

$(2ab - a^2\lambda)(1+\lambda) - b = 0$ ………………………………………………………………………..(3)

Adding (2) and (3), We get

$(2ab - a^2\lambda)(1+\lambda-\lambda) + a^2(1+\lambda) + b^2 - a - b = 0$

$\Rightarrow 2ab - a^2\lambda + a^2 + a^2\lambda + b^2 - a - b = 0$

$\Rightarrow a^2 + 2ab + b^2 - (a+b) = 0$

$\Rightarrow (a+b)^2 - (a+b) = 0$

$\Rightarrow (a+b)(a+b-1) = 0$

$\Rightarrow$Either $(a+b) = 0$ or $(a+b) = 1$

So for projection, the above two cases will be considered

Case (1): let $(a+b)=1$ then $b=1-a$

Putting the value of b = 1 - a in equation (2) , we get

$$a^2(1+\lambda+\lambda^2)-2a(1-a)\lambda-a+(1-a)^2=0$$

$\Rightarrow a^2(\lambda^2+3\lambda+2)-a(3+2\lambda)+1=0$

$\Rightarrow a^2(\lambda+1)(\lambda+2)-a(\lambda+1+\lambda+2)+1=0$

$\Rightarrow$ [a($\lambda+1$)-1] [a($\lambda+2$) $-$ 1]=0

$\Rightarrow a=\frac{1}{\lambda+1}\ or\ \frac{1}{\lambda+2}$

Then $b=\frac{\lambda}{\lambda+1}$ or $\frac{\lambda+1}{\lambda+2}$

Hence corresponding projections are

$\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}\ and\ \frac{E^2}{\lambda+2}+\frac{(\lambda+1)E}{\lambda+2}$

Case (2) :- Let a + b = 0 or b = $-a$

So from Equation (2)

$$a^2(1+\lambda)+a^2-a-\lambda\{-2a^2-a^2\lambda\}=0$$

$\Rightarrow a^2[\lambda^2+3\lambda+2]-a=0$

$\Rightarrow a[a(\lambda+1)(\lambda+2)-1]=0$

$\Rightarrow a=\frac{1}{(\lambda+1)(\lambda+2)}$; (Assuming $a\neq 0$)

Therefore, $b=\frac{-1}{(\lambda+1)(\lambda+2)}$

Hence the corresponding projection is

$$\frac{E^2-E}{(\lambda+1)(\lambda+2)}$$

So in all we get three projections. Call them A, B & C.

i.e. A=$\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}$, B = $\frac{E^2-E}{(\lambda+1)(\lambda+2)}$

and C = $\frac{E^2}{1+\lambda}+\frac{\lambda E}{1+\lambda}$.

3.2 Relation between A,B and C

A+B = $\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}+\frac{E^2}{(\lambda+1)(\lambda+2)}-\frac{E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+1)+(\lambda+1)^2E+E^2-E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+2)+\lambda(\lambda+2)E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2+\lambda E}{\lambda+1}$

$=\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}=c$

Hence $(A+B)^2=C^2\Rightarrow A^2+B^2+2AB=C^2$

$\Rightarrow A+B+2AB=C$

$\Rightarrow 2AB=0(Since\ \ A+B=C)$

$\Rightarrow AB=0$

Let $\mu=\lambda+1$

Then A $=\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}, B=\frac{E^2-E}{\mu(\mu+1)}$ and $C=\frac{E^2}{\mu}+\frac{(\mu-1)E}{\mu}$

Also $A-\mu\beta=\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}-\frac{E^2}{\mu+1}+\frac{E}{\mu+1}$

$=\frac{\mu E+E}{\mu+1}\ =\ \frac{(\mu+1)E}{\mu+1}=E$

Thus $E=A-\mu B$.

3.3 On ranges and null spaces of $\lambda - jection$

We show that

$R_E = R_C \ and \ N_E = N_C$

Where R_E stands for range of operator E and N_E for Null Space of E and similar notations for other operators.

Let $x \in R_E \ then \ x = Ez \ \ for \ \ some \ z \ in \ X.$

Therefore,

$$Cx = CEz = \frac{\{(\mu-1)E+E^2\}Ez}{\mu}$$
$$= \frac{\{(\mu-1)E^2+E^3\}\,Z}{\mu}$$
$$= \frac{\{(\mu-1)E^2+\mu E-(\mu-1)E^2\}\,Z}{\mu} \qquad \text{(Since } E^3 = (1+\lambda)E - \lambda E^2\text{)}$$
$$= \left(\frac{\mu E}{\mu}\right)z = Ez = x$$

Thus $Cx = x \Rightarrow x \in R_C$

Therefore $R_E \subseteq R_C$

Again if $x \in R_C \ then \ x = Cx = \left(\frac{E^2+(\mu-1)E}{\mu}\right)x$

$$= \frac{E(E+\lambda I)x}{\lambda+1} \in R_E$$

Hence $R_C \subseteq R_E$

Therefore $R_E = R_C$

Now , z$\in N_E \Rightarrow Ez = 0$

$\Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$

$\Rightarrow Cz = 0$

$\Rightarrow$ z $\in N_C$

Therefore , $N_E \subseteq N_C$

Also if z$\in N_C \Rightarrow \ Cz = 0 \Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$

$$\Rightarrow E\left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$$
$$\Rightarrow \left(\frac{E^3+(\mu-1)E^2}{\mu}\right)z = 0$$
$$\Rightarrow \left(\frac{\mu E}{\mu}\right)z = 0 \Rightarrow Ez = 0 \ \Rightarrow z \in N_E$$

Thus $N_c \subseteq N_E$,

Therefore , $N_E = N_C$

Now we show that

$\boldsymbol{R_A = \{z: Ez = z\} \ and \ R_B = \{z: Ez = -\mu z\}}$

Since A is a Projection ,

$R_A = \{z: Az = z\}$

Let $z \in R_A$. Then $Ez = EAz = E\left(\frac{E^2+\mu E}{\mu+1}\right)z$

$$= \left(\frac{E^3+\mu E^2}{\mu+1}\right)z$$
$$= \left(\frac{E^3+(\mu-1)E^2+E^2}{\mu+1}\right)z$$
$$= \left(\frac{\mu E+E^2}{\mu+1}\right)z$$
$$= Az = z$$

Thus $R_A \subseteq \{z: Ez = z\}$

Conversely , Let $Ez = z \ then \ E^2z = z$

So $Az = \left(\frac{E^2+\mu E}{\mu+1}\right)z = \frac{z+\mu z}{\mu+1} = z \Rightarrow z \in R_A$

Hence $\{z: Ez = z\} \subseteq R_A$
Therefore, $R_A = \{z: Ez = z\}$
Next we show that
$R_B = \{z: Ez = -\mu z\}$
Since B is a Projection,
$R_B = \{z: Bz = z\}$
Let $Ez = -\mu z \; then \; E^2 z = \mu^2 z$
Hence $\left(\frac{E^2-E}{\mu(\mu+1)}\right) z = \frac{\mu^2 z + \mu z}{\mu(\mu+1)} = \frac{\mu(\mu+1)}{\mu(\mu+1)} z = z$
i.e. $Bz = z$ $(since \; B = \frac{E^2-E}{\mu(\mu+1)})$
$\Rightarrow z \in R_B$
$Therefore, \{z: Ez = -\mu z\} \subseteq R_B$
Conversely, let $z \subseteq R_B. Then \; Bz = z$
Hence $Ez = EBz = E\left(\frac{E^2-E}{\mu(\mu+1)}\right) z$

$$= \left(\frac{E^3-E^2}{?(?+1)}\right) z$$

But $E^3 - E^2 = \mu E - (\mu - 1)E^2 - E^2$

$$= \mu E - \mu E^2 = \mu(E - E^2)$$

So $Ez = \frac{\mu(E-E^2)z}{\mu(\mu+1)} = \frac{-\mu(E^2-E)z}{\mu(\mu+1)}$

$$= -\mu Bz = -\mu z$$

$So \; R_B \subseteq \{z: Ez = -\mu z\}$
Therefore $R_B = \{z: Ez = -\mu z\}$

Now we show that $R_A \cap R_B = \{0\}$
Let $z \in R_A \cap R_B$
Then $z \in R_A \; and \; z \in R_B$
If $z \in R_A \; than \; Ez = z$
If $z \in R_B \; than \; Ez = -\mu z$
Thus $Ez = z = -\mu z$
$\Rightarrow \mu z + z = 0 \Rightarrow (\mu + 1)z = 0 \Rightarrow z = 0 \; (since \; \mu + 1 \neq 0)$
Therefore, $R_A \cap R_B = \{0\}$

Theorem (1): If $E_1 \; and \; E_2 \; are \; \lambda - jections \; on \; a \; linear \; space \; X \; and \; E_1 E_2 = E_2 E_1 = 0 \; then \; E_1 + E_2 \; is \; also \; a \; \lambda - jection$ such that $N_{E_1+E_2} = N_{E_1} \cap N_{E_2}$ and $R_{E_1+E_2} = R_{E_1} \oplus R_{E_2}$

Proof :- Since $E_1 E_2 = E_2 E_1 = 0$,
$(E_1 + E_2)^2 = E_1^2 + E_2^2$ and $(E_1 + E_2)^3 = E_1^3 + E_2^3$
$So \; (E_1 + E_2)^3 + \lambda(E_1 + E_2)^2 = E_1^3 + E_2^3 + \lambda(E_1^2 + E_2^2)$

$$= E_1^3 + \lambda E_1^2 + E_2^3 + \lambda E_2^2$$
$$= (1 + \lambda)E_1 + (1 + \lambda)E_2$$
$$= (1 + \lambda)(E_1 + E_2)$$

Hence $E_1 + E_2 \; is \; a \; \lambda - jection$
Let A_1, B_1, C_1 respectively projections in case of E_1; A_2, B_2, C_2 in case of $E_2 \; and \; A, B, C \; in \; case \; of \; E_1 + E_2$
$Also \; C_1 + C_2 = \frac{E_1^2}{\mu} + \frac{(\mu-1)E_1}{\mu} + \frac{E_2^2}{\mu} + \frac{(\mu-1)E_2}{\mu}$

$$= \frac{1}{\mu}(E_1^2 + E_2^2) + \left(\frac{\mu-1}{\mu}\right)(E_1 + E_2) = C$$

Thus $R_{E_1+E_2} = R_C = R_{C_1+C_2}$
Now we show that $R_{C_1+C_2} = R_{C_1} + R_{C_2}$
Let $z \epsilon R_{C_1+C_2}$ $then\ z = (C_1 + C_2)z_1 for\ some\ z_1\ in\ X$
$= C_1 z_1 + C_2 z_1 \epsilon R_{C_1} + R_{C_2}$
$So\ R_{C_1+C_2} \subseteq R_{C_1} + R_{C_2}$
Conversely, $let\ z \epsilon R_{C_1} + R_{C_2}\ then\ z = z_1 + z_2$; where $z_1 \epsilon R_{C_1}, z_2 \epsilon R_{C_2}$
Hence $C_1 z_1 = z_1\ and\ C_2 z_2 = z_2$
But $C_2 z_1 = C_2 C_1 z_1 = 0$ ($Since\ E_1 E_2 = E_2 E_1 = 0$)
Similarly $C_1 z_2 = C_1 C_2 z_2 = 0$
Hence $z = z_1 + z_2 = C_1 z_1 + C_2 z_2 + C_1 z_2 + C_2 z_1$
$= (C_1 + C_2)(z_1 + z_2) = (C_1 + C_2)z$
Thus $z \in R_{C_1+C_2}$
Therefore, $R_{C_1} + R_{C_1} \subseteq R_{C_1+C_2}$
Hence $R_{C_1+C_2} = R_{C_1} + R_{C_2}$
If $z \in R_{C_1} \cap R_{C_2}$ $then\ z \in R_{C_1} and\ z \in R_{C_2}$
If $z \in R_{C_1}$ $then\ C_1 z = z\ and$ if $z \in R_{C_2}$ $then\ C_2 z = z$
Now $C_1 z = z = C_2 z = C_2(C_1 z) = (C_2 C_1)z = 0$
$\Rightarrow z = 0$
Therefore $R_{C_1} \cap R_{C_2} = \{0\}$
Hence $R_{C_1+C_2} = R_{C_1} \oplus R_{C_2}$
So finally , We have

$$R_{E_1+E_2} = R_C = R_{C_1+C_2} = R_{C_1} \oplus R_{C_2} = R_{E_1} \oplus R_{E_2}$$

Now we prove
$\boldsymbol{N_{E_1+E_2} = N_{E_1} \cap N_{E_2}}$
Or $\boldsymbol{N_{C_1+C_2} = N_{C_1} \cap N_{C_2}}$
Let $z \in N_{C_1} \cap N_{C_2} \Rightarrow C_1 z = 0 = C_2 z$
$\Rightarrow (C_1 + C_2)z = 0$
$\Rightarrow z \in N_{C_1+C_2}$
Therefore $N_{C_1} \cap N_{C_2} \subseteq N_{C_1+C_2}$
Conversely , let $z \in N_{C_1+C_2} \Rightarrow (C_1 + C_2)z = 0$
$\Rightarrow C_1 z + C_2 z = 0$
$\Rightarrow C_1 z = -C_2 z$
Hence $C_1 z = C_1^2 z = C_1(C_1 z) = C_1(-C_2 z) = -C_1 C_2 z = 0$
$\Rightarrow z \in N_{C_2}$
Also $C_2 z = C_2^2 z = C_2(C_2 z) = C_2(-C_1 z) = -C_2 C_1 z = 0$
$\Rightarrow z \in N_{C_2}$
So $z \in N_{C_1} \cap N_{C_2}$
Therefore , $N_{C_1+C_2} \subseteq N_{C_1} \cap N_{C_2}$
Hence $N_{C_1+C_2} = N_{C_1} \cap N_{C_2}$
Or , $N_{E_1+E_2} = N_{E_1} \cap N_{E_2}$

Theorem (2): If $E_1\ and\ E_2\ are\ \lambda - jections\ \ on\ a\ linear\ space\ X$,then

$$C_1 E_2 = C_2 C_1 E_1 \Leftrightarrow R_{E_2} = R_{E_2} \cap R_{E_1} \oplus R_{E_2} \cap N_{E_1}$$

Proof : Let $R_{E_2} = R_{E_2} \cap R_{E_1} \oplus R_{E_2} \cap N_{E_1}$
Let $z \in X\ than\ E_2 z \in R_{E_2}. Let\ E_2 z = z_1 + z_2$;
where $z_1 \in R_{E_2} \cap R_{E_1}$, $z_2 \in R_{E_2} \cap N_{E_1}$

Now, $z_1 \in R_{E_2} \cap R_{E_1} \Rightarrow z_1 \in R_{E_2}\ and\ z_1 \in R_{E_1}$

$$\Rightarrow E_2 z_1 = z_1 and\ E_1 z_1 = z_1$$

$\Rightarrow C_2 z_1 = z_1 and\ C_1 z_1 = z_1$ (Since $R_E = R_C$)

Therefore , $z_1 = C_1 z_1 = C_1(C_2 z_1) = C_1 C_2 z_1$

$$z_2 \in R_{E_2} \cap N_{E_1} \Rightarrow z_2 \in R_{E_2}\ and\ z_2 \in N_{E_1}$$

$z_2 \in R_{E_2} \Rightarrow C_2 z_2 = z_2, z_2 \in N_{E_1} \Rightarrow C_1 z_2 = 0$ (since $N_E = N_C$)

So $C_1(E_2 z) = C_1(z_1 + z_2) = C_1 z_1 + C_1 z_2 = z_1 + 0 = z_1$

Also $C_2 C_1 E_2 z = C_2 z_1 = z_1$

Therefore , $C_1 E_2 z = C_2 C_1 E_2 z\ for\ any\ z\ in\ X$

Hence $C_1 E_2 = C_2 C_1 E_2$

Conversly , let $C_1 E_2 = C_2 C_1 E_2$, $than\ \ C_1 E_2^2 = C_2 C_1 E_2^2$

Hence $C_2 C_1 C_2 = C_2 C_1 \left\{\frac{(\mu-1)E_2 + E_2^2}{\mu}\right\}$

$$= \left(\frac{\mu-1}{\mu}\right) C_2 C_1 E_2 + \frac{1}{\mu}\ C_2 C_1 E_2^2$$

$$= \left(\frac{\mu-1}{\mu}\right) C_1 E_2 + \frac{1}{\mu}\ C_1 E_2^2 \quad (\text{Since} C_1 E_2 = C_2 C_1 E_2\ )$$

$$= C_1 \left\{\left(\frac{\mu-1}{\mu}\right) E_2 + \frac{1}{\mu}\ E_2^2\right\} = C_1 C_2$$

Let z $\in R_{E_2}$, $than\ C_2 z = z. Let\ y = C_1 z \in R_{E_1}$

Also $y = C_1 z = C_1 C_2 z = C_2 C_1 C_2 z \in R_{E_2}$

So y $\in R_{E_2} \cap R_{E_1}$

Also , $C_2(z - y) = C_2(z - C_1 z) = C_2(z - C_1 C_2 z)$

$$= C_2 z - C_2 C_1 C_2 z = C_2 z - C_1 C_2 z\ = z - C_1 z = z - y$$

Hence $z - y \in R_{E_2}$

Also $C_1(z - y) = C_1(z - C_1 z) = C_1 z - C_1^2 z = C_1 z - C_1 z = 0$

$\Rightarrow z - y \in N_{E_1}$

Hence $z - y \in R_{E_2} \cap N_{E_1}$. So z = $y + z - y \in R_{E_2} \cap R_{E_1} + R_{E_2} \cap N_{E_1}$

$\Rightarrow R_{E_2} \subseteq R_{E_2} \cap R_{E_1} + R_{E_2} \cap N_{E_1}$

Obviously , $R_{E_2} \cap R_{E_1} + R_{E_2} \cap N_{E_1} \subseteq R_{E_2}$

Hence $R_{E_2} = R_{E_2} \cap R_{E_1} + R_{E_2} \cap N_{E_1}$

Also, $R_{E_2} \cap R_{E_1} \cap R_{E_2} \cap N_{E_1} = R_{E_2} \cap R_{E_1} \cap N_{E_1} = R_{E_2} \cap \{0\} = \{0\}$

Hence $R_{E_2} = R_{E_2} \cap R_{E_1} \oplus R_{E_2} \cap N_{E_1}$

Theorem (3) : If E_1, E_2 are two $\lambda - jections\ \ on\ linear\ space\ X, then$

$$E_2 C_1 = E_2 C_1 C_2 \Leftrightarrow N_{E_2} = N_{E_2} \cap R_{E_1} \oplus N_{E_2} \cap N_{E_1}$$

Proof: Let $N_{E_2} = N_{E_2} \cap R_{E_1} \oplus N_{E_2} \cap N_{E_1}$

Let $z \in X$, $then\ \ E_2(I - C_2)z = (E_2 - E_2 C_2\ )z = (E_2 - E_2)z = 0$

Hence $(I - C_2)z \in N_{E_2}$

Let $(I - C_2)z = z_1 + z_2$; $\ where\ z_1 \in N_{E_2} \cap R_{E_1} and\ \ z_2 \in N_{E_2} \cap N_{E_1}$

Hence $E_1 z_2 = 0, E_2 z_1 = 0, E_2 z_2 = 0, C_1 z_1 = z_1$

Now $E_2 C_1(I - C_2)z = E_2 C_1(z_1 + z_2) = E_2(C_1 z_1 + C_1 z_2)$

$= E_2(z_1 + C_1 z_2) = E_2 z_1 + E_2 C_1 z_2 = E_2 C_1 z_2 = E_2\ \left(\frac{E_1^2 + (\mu-1)E_1}{\mu}\right) z_2$

$$= \frac{E_2 {E_1}^2 z_2}{\mu} + \left(\frac{\mu-1}{\mu}\right) E_2 E_1 z_2 = 0$$

Hence for any z, $E_2 C_1(I - C_2)z = 0$

Therefore ,$E_2 C_1(I - C_2) = 0 \Rightarrow E_2 C_1 = E_2 C_1 C_2$

Conversely , let the above relation hold

Let $z \in N_{E_2}$ $then\ E_2 z = 0\ then\ \ C_2 z = \left(\frac{E_2^2 + (\mu - 1)E_2}{\mu}\right) z = 0$

$Since\ E_1(I - C_1)z = (E_1 - E_1 C_1)z = 0\ , \Rightarrow (I - C_1)z \in N_{E_1}$

$$Also, E_2(I - C_1)z = E_2 z - E_2 c_1 z = E_2 z - E_2 c_1 c_2 z$$

$$= 0 - E_2 c_1 c_2 z = 0 (since\ E_2 z = 0 \Rightarrow C_2 z = 0)$$

$$Therefore\ , (I - C_1)z \in N_{E_2}\ Hence\ (I - C_1)z \in N_{E_1} \cap N_{E_2}$$

Also $E_2 c_1 z = E_2 c_1 c_2 z = E_2 c_1 (c_2 z) = 0$

Hence $c_1 z \in N_{E_2}$. Also $C_1 z \in R_{E_1}$

Hence $C_1 z \in N_{E_2} \cap R_{E_1}$

$so\ z = (I - C_1)z + c_1 z \in N_{E_2} \cap N_{E_1} + N_{E_2} \cap R_{E_1}$

$Therefore\ , N_{E_2} \subseteq N_{E_2} \cap N_{E_1} + N_{E_2} \cap R_{E_1}$

But $N_{E_2} \cap N_{E_1} + N_{E_2} \cap R_{E_1} \subseteq N_{E_2}$

$So\ N_{E_2} = N_{E_2} \cap R_{E_1} + N_{E_2} \cap N_{E_1}$

Also , $N_{E_2} \cap R_{E_1} \cap N_{E_2} \cap N_{E_1} = N_{E_2} \cap \left(R_{E_1} \cap N_{E_1}\right) = N_{E_2} \cap \{0\} = \{0\}$

$So\ N_{E_2} = N_{E_2} \cap R_{E_1} \oplus N_{E_2} \cap N_{E_1}$

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