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A Comparison of MEWMA and Hotelling's T^2 Control Charts Procedures with Industrial Application

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ABSTRACT

In this paper, comparing between two important procedures: Multivariate Exponentially Weighted Moving Average (MEWMA)quality control chart and Hotelling's T^2quality control chart. The first procedure MEWMA is an example of a multivariate charting scheme whose monitoring statistic is unable to determine which variable caused the signal. The average run length (ARL) performance of MEWMA chart is studied. The second procedure is Hotelling's T^2quality control chart is used to determine whether or not the process mean vector for two or more variables is in-control. It is allowing us to simultaneously monitor whether two or more related variables are in control, and it is shown that multivariate quality control chart do not indicate which variables cause the out-of-control signal so that the interpretation of the out-of-control signal. Also, in this paper develops the multivariate quality control charts of the out of-control signal, this will be maintained by making and industrial application on the fertilizers factory. It is important to know that this factory has a special department for quality control, in order to maintain International Standardization Organization (ISO).

***KEY WORDS**: Quality Control, Multivariate Statistical Analysis, Average Run Length Performance, Monitoring Process.*

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I. INTRODUCTION

The statistical control chart is a well-known tool in today's industry, and it is One of the most powerful tools in quality control. First developed in the 1920's by Walter Shewhart, the control chart found widespread use during World War II and has been employed, with various modifications ever since. The drawbacks to multivariate charting schemes is their inability to identify which variable was the source of the signal.

With today's use of computers, it is common to monitor several correlated quality characteristics simultaneously. Various types of multivariate control charts have been proposed to take advantage of the relationships among the variables being monitored. Alt (1985), Jackson (1988), Lowry and Montgomery (1995), and Mason et al. (2002) discuss much of the literature on this topic. The formatter will need to create these components, incorporating the applicable criteria that follow.

The rapid growth data acquisition technology and the uses of online computers for process monitoring led to an increased interest in the simultaneous control of several related quality characteristics. These techniques are often referred to multivariate statistical process control procedures. The use of separate univariate control chart for each quality characteristic has proved to be inappropriate. This is because, it neglects the correlation between the multiple quality characteristics; and this leads to incorrect results.

The modern statistical process control took place when Walter A. Shewhart (1931) developed the concept of a control chart based on the monitoring of the process mean level through sample mean ($\bar{X}$ chart) and process dispersion through sample range (R chart) or sample standard deviation chart. In the multivariate setting, Harold Hotelling (1947) published what can be called the first major works in multivariate quality control. Hotelling developed the T^2 statistic and the statistics based on the sample variance-covariance matrix S procedure, and its extensions to control charts to combine measurements taken on variables in several dimensions into a single measure of excellence.

After Hotelling there was no significant work done in this field until the early sixties, when with the advances in computers, interest in multivariate statistical quality control was revived. Since then, some authors have done some work in this area of multivariate quality control.

Houshmand, A. and Javaheri, A. (1998) presented two procedures to control the covariance matrix in a multivariate setting. The advantages of these procedures are that they allow the investigators to identify the sources of the out-of-control signal. These procedures are based on constructing tolerance regions to control the parameters of the correlation matrix.

Linna, Woodall (2001) presented a model for correlated quality variables with measurement error. The model determined the performance of the multivariate control charting methods. The usual comparison of control chart performance does not directly Apply in the presence of measurement error.

The most familiar multivariate process monitoring and control procedure is the Hotelling's T^2 control chart for monitoring the mean vector of the process. It is a direct analog of the univariate Shewhart $\bar{X}$ chart. Shewhart, a pioneer in the development of the statistical control chart (Shewhart charts), first recognized the need to consider quality control problems as multivariate in character. Hotelling (1947) did the original work in multivariate quality control. He applied his procedure, which assumed that P-quality characteristics are jointly distributed as P-variate normally and random samples of size n are collected across time from the process. T^2 is sensitive to shifts in the means, as well as to shifts in the variance, but it cannot distinguish between location shifts and scale shifts.

Multivariate charts are also useful for monitoring quality profiles as discussed by Woodall et al. (2004). Alt (1985) defined two phases in constructing multivariate control charts, with Phase I divided into two Stages. In the retrospective Stage 1 of Phase I, historical data (observations) are studied for determining whether the process was in control and to estimate the in-control parameters of the process. The Hotelling's T^2 control chart is utilized in this stage (Alt and Smith, 1988, Tracy et al. 1992, and Wierda,1994). In Phase II, control charts are used with future observations for detecting possible departure from the process parameters estimated in Phase I. In Phase II, one uses charts for detecting any departure from the parameter estimates, which are considered in the in-control process parameters (Vargas,2003).

An important aspect of the Hotelling's T^2 control chart is how to determine the sample variance-covariance matrix used in the calculation of the chart statistics, the upper control limit (UCL) and the lower control limit (LCL).

Onwuka (2012) discussed the Principal Component Analysis and Hotelling's T^2 tests were used, with the 3-characteristics measured showing negligible low correlation with nearly all the correlation coefficients small.

II. MULTIVARIATE QUALITY CONTROL CHART

Multivariate quality control charts are a type of variables control that how correlated, or dependent, variables jointly affect a process or outcome. The multivariate quality control charts are powerful and simple visual tools for determining whether the multivariate process is in-control or out-of-control. In other words, control charts can help us to determine whether the process average (center) and process variability (spread) are operating at constant levels. Control charts help us focus problem – solving efforts by distinguishing between common and assignable cause variation. Multivariate control chart plot statistical from more than one related measurement variable. The multivariate control chart shows how several variables jointly influence a process or outcome.

It is demonstrated that if the data include correlated variables the use of separate control chart is misleading because the variables jointly affect the process. If we use separate univariate control chart in a multivariate situation, type I error and probability of a point correctly plotting in- control are not equal to their expected values the distortion of those values increases with the number of measurement variables.

It is shown that multivariate control chart has several advantages in comparison with multiply univariate charts:

- The actual control region of the related variables is represented.
- We can maintain specification type I error.
- A signal control limit determines whether the process is in control.
- Multivariate control chart simultaneously monitors two or more correlated variables. To monitor more than one variable using univariate charts, we need to create a univariate chart for each variable.
- The scale on multivariate control charts unrelated to the scale of any of the variables.
- Out-of-control signals in multivariate charts do not reveal which variable or combination of variables cause the signal. A multivariate control chart consists of:
- Plotted points, each for which represents a rational subgroup of data sampled from the process, such as a subgroup mean vector individual observation, or weighted statistic.

- A center line, which represents the expected value of the quality characteristics for all subgroups.
- Upper and lower control limits (UCL and LCL), which are set a distance above and below the center line. These control limits provide a visual display for the expected amount for variation. The control limits are based on the actual behavior of the process, not the desired behavior or specification limits. A process can be in control and yet not be capable of meeting requirements.

III. COMPARISON METHODOLOGY

The following arrangement used for compare methodology between two procedures:

3.1 The MEWMA Control Chart

3.1.1 The EWMA Control Chart Procedure

Timm (1996) developed a control chart using the EWMA to control a process mean. The EWMA techniques give the most recent observation the greatest weight with all previous observations weights decreasing in a geometric (exponential) progression from the recent back to the first. To demonstrate the EWMA technique, suppose that we observe sample means $X_1 . X_2 . X_3 , \dots$ in the univariate case, where,

$$X \sim N\ (\mu_0 . \sigma_x^2) \tag{1}$$

MacGregor (1995) as introduced the univariate Exponentially Weighed Moving Average (EWMA) control chart:

$$Z_i = rX_i + (1-r)Z_{i-1} . \quad i = 1.2.3. \dots \tag{2}$$

where $r$is a smoothing constant and Z_i is the value of the EWMA after observation i. where i represents the observation number as well as an index of a point in time, where we required $Z_0 = \mu_0 = 0$ without loss of generality and $0 < r \leq 1$. and he supposed that if $X_1 . X_2 . X_3 , \dots$ are independently and identically distributed $N\ (0 . \sigma^2)$ random variables, the mean ofZ_i is0 and the variance is

$$var\ (Z_i) = \sigma_Z^2 = \left\{\frac{r\,[1-(1-r)^{2i}]}{2-r}\right\}\sigma_X^2 . \quad i = 1.2.3. \dots \tag{3}$$

Thus, he suggested that when the in-control value of the mean is 0, the control limits of the EWMA chart are at $\pm L\sigma_{Zi}$. where L and r are the parameters of the chart.

Lucas and Saccucci (1990) have discussed that the choice of L and r for the interval $(0.05 \leq r \leq 0.25)$ work well in practice, with $r = 0.05$. $r = 0.10$ and $r = 0.20$ being popular choices. A good rule of thumb is to use smaller values of r to detect smaller shifts. They have also found that $L = 3$ (the usual three – sigma limits) works although when r is small – say, $r \leq 0.1$ - there is an advantage in reducing the width of the limits by using a value of L between about 2.6 and 2.8 although their control limits, were based on the asymptotic form of σ_{Zi} which is given by $\sigma_{Zi} \cong \sigma_X \sqrt{\frac{r}{2-r}}$.

3.1.2 Multivariate Extension of The EWMA Control Chart

Lowry et al. (1992) has generalized the concept of the univariate EWMA control chart to the multivariate case. They defined the MEWMA vectors as:

$$Z_i = RX_i + (1-R)Z_{i-1} . \quad i = 1.2.3. \dots \tag{4}$$

where $Z_0 = 0$ and

$R = diag\ (r_1 . r_2 . \dots . r_p .\ 0 < r_j \leq 1) . j = 1.2. \dots . p$ and $p > 1$.

The MEWMA control chart gives an out of control signal as soon as:

$$T_i^2 = Z_i^{`} \textstyle\sum_i^{-1}\ Z_i > L \tag{5}$$

where $L > 0$ is chosen to achieve a specified in-control ARL, and Σ_{Zi} is the covariance matrix of Z_i .

Mason et al. (2002) provide a technique to interpret the out-of-control signal and identify an assignable cause from a multivariate control chart. Their method can be applied to MEWMA control chart.

Mason et al.'s method has assumed that Phase I of process control is when the first subgroups are drawn from the process and little is known about the joint distribution of the quality characteristics. Phase II is when sufficient information is known about the process and subgroups are drawn to test if the process is in control.

Often there is no reason to apply different exponential weights to past observations of the p deferent quality characteristics. In this situation, Lowry et. (1992) assumed the equal weights across characteristics where$r = r_j$. $j = 1. \dots, p,$ the MEWMA vectors can then be written as

$$Z_i = rX_i + (1-r)Z_{i-1} . \quad i = 1.2.3. \dots \tag{6}$$

Under the assumption of equal weights, Lowry et al. (1992) have shown that the covariance matrix of Z_i can be written in terms of the exponential weight r and the covariance matrix of the process data $\Sigma_X$as:

$$\Sigma_{Zi} = \left\{\frac{r\,[1-(1-r)^{2i}]}{2-r}\right\}\Sigma_X \tag{7}$$

note that if $r = 1$, the MEWMA chart is equivalent to Hotellin's T^2 chart.

Lowry et al. (1992) have suggested that when the process is likely to stay in-control for some time period, the asymptotic from of the covariance matrix Σ_X used to calculate the MEWMA test statistic:

$$\Sigma_{Zi} = \left\{\frac{r}{2-r}\right\}\Sigma_X \tag{8}$$

Lowry et al. (1992) have also, added the use of exact variance of the EWMA statistic leads to a natural fast initial response for the EWMA chart. Thus, the initial that, out-of-control conditions can also, applied to the MEWMA chart. Because, however, it may be more likely that the process will stay in control for a while and then shift out of control, they assumed for chart design that the asymptotic (as $i \to \infty$) covariance matrix, is given by Equation (8).

Linderman and Love (2000) have pointed out that the used of exact covariance matrix and the asymptotic covariance matrix lead to two different procedures; actually, they have concentrated their efforts concerned solely on the MEWMA chart using the asymptotic covariance matrix and the exact covariance matrix.
They have depended on the use of MEWMA chart in order to study the p quality characteristics associated with a process. The process begins in the in-control state with mean vector $\mu_0 = 0$ and covariance matrix Σ_X . They supposed also that the process is subject to a single assignable cause, which shifts the process mean from μ_0 to a point on the constant probability density contour *D*; defined by

$$D = \lfloor \mu_1 | \acute{\mu}_1 \textstyle\sum_X^{-1} \mu_1 = \delta^2 \rfloor \tag{9}$$

where δ, the parameter describing the size of shift, is known. Note that Equation (9) is the constant probability density contour for a p- dimensional multivariate normal distribution. This contour forms an ellipsoid which is centered at μ_0 and has axes at

$$\pm\, \delta \sqrt{\xi_j e_j}\ \text{, where } \textstyle\sum_X e_j = \xi_j e_j\ \text{. for } j = 1. \dots, p.$$

Khoo (2003) has suggested that the main choice for the value of r in Equations (7) and (8) is, founded on the magnitude of shift where a quick detection is required. In general, small values of r used for quick detection of small shifts, whereas larger values of r used for quick detection of large shifts. He has, also added that in case of ($i \to \infty$) the exact covariance matrix in Equation (7) is approximately equal to the asymptotic covariance matrix in Equation (8); pointing out that this, the two covariance matrices in Equation (7) and (8) differ only in initial periods when i values are small. However, the advantages of the exact MEWMA chart over the asymptotic MEWMA chart are that the exact MEWMA chart enables quicker detection of initial out-of-control conditions and is more sensitive for detecting shifts involving smaller values of r.

3.1.3 The Average Run Length Performance for a MEWMA

The average run length is metric used to determine the control chart's ability in order to determine if the process is in control or out of control. Lowry et al. (1995) established that ARL performance for a MEWMA control chart is directionally invariant and determined solely by the non-centrality parameter. This means that the average run length depends only on the distance between the in control and out of control mean, and not on the direction. The non-centrality parameter given by Lowry et al. (1995), as:

$$\delta^2(\mu_y) = (\mu_y - \mu_0)' \textstyle\sum_X^{-1} (\mu_y - \mu_0) \tag{10}$$

where μ_y is the mean when the process is out of control. They supposed that if $\mu_i = \mu$, $i = 1.2. \dots$ the non-centrality parameter is given as:

$$\delta^2 = \left(\mu' \sum\nolimits_X^{-1} \mu\right)$$

Recent researches have been developed to approximate the ARL for a MEWMA control chart. These methodologies have assumed that the in-control distribution is given as $N_p(\mu_0.\Sigma_X)$. In addition, all these methodologies have made use of the ARL following fact that the performance depends only on the non-centrality parameter. As a result, to determine the ARL for a MEWMA control chart with an in-control distribution $N_p(\mu_0.\Sigma)$ and out-of-control distribution $N_p(\mu_1.\Sigma)$, we can determine the ARL performance of a MEWMA control chart with an in-control distribution $N_p(0.1)$ and out of control distribution $N_p(\mu_\delta.\mathrm{I})$, where $\mu_\delta = (\delta.0.\dots.0)$. That is, the non-centrality parameter between both problems is identical.
Actually, the ARL performance, which only depends on the non-centrality parameter, is central to methodologies that have suggested to approximate ARL0 and ARL1 for a MEWMA chart. The approximation methodologies include simulation Lowry et al. (1995), a bivariate Markov chain method Runger et al. (1996), and integral equation Rigdon (1995).

3.2 Construction of Hotelling's T^2 Control Chart

The Hotelling multivariate control chart signals that a statistically significant shift in the mean has occurred as soon as:

$$\chi^2 = (\bar{X}_i - \mu_o)'\Sigma^{-1}(\bar{X}_i - \mu_o) \tag{11}$$

If the sample covariance matrix Σ and the sample mean vector μ_0 are known, but if Σ and μ_0 are known, then the T^2 statistic is the appropriate statistic for the Hotelling multivariate control chart. In this case the sample covariance matrix, *S* and sample mean vector $\bar{X}$, are used to estimate Σ andμ_0 respectively.

This statistic has the from:

$$T^2 = (\bar{X}_i - \bar{\bar{X}})'S^{-1}(\bar{X}_i - \bar{\bar{X}}) \tag{12}$$

Suppose that we have a random sample from a multivariate normal distribution – Say, $X_1 . X_2 . X_3$, …, X_n where the i^{th} sample vector contains observations, $X_{i1} . X_{i2} . X_{i3}$, …, X_{ip} .

Let the sample mean vector is:

$$\underline{\bar{X}}_{1\times p} = (\bar{X}_1 . \bar{X}_2 . \dots . \bar{X}_p)$$

where

$$\bar{X}_i = \sum_{L=1}^{n} X_{iL} \qquad (i = 1.2. \dots . p)$$

and the sample covariance matrix is:

$$S = \begin{bmatrix} s_1^2 & s_{12} & \cdots & s_{1p} \\ s_{21} & s_2^2 & \cdots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & \cdots & s_p^2 \end{bmatrix} \tag{13}$$

where s_i^2 is the variance of the i^{th} variable and $(ij)^{th}$element of *S*-matrix is the estimated covariance between the variables *i* and *j*,

$$S_{ij} = \frac{1}{n-1}\sum_{L=1}^{n}(X_{iL} - \bar{X}_i)(X_{jL} - \bar{X}_j)$$

Not that we can show that the sample mean vector and the sample covariance matrix are unbiased estimators of the corresponding population quantities that is

$$E(\underline{\bar{X}}) = \underline{\mu} \quad and \quad E(S) = \Sigma \tag{14}$$

Seber (1984) gives the distribution properties of this estimate as follows:

(i) $\bar{X} \sim N_p\left(\mu . \frac{1}{n}\Sigma\right)$. (15)

(ii) If $\bar{X}$ distribution as in (i) then:$n(\bar{X} - \mu)'\Sigma^{-1}(\bar{X} - \mu) \sim \chi^2_p$ (16)

(iii) $(n-1)S \sim W_p(n - 1.\Sigma)$

where $W_p(n - 1.\Sigma)$ stands for the Wishart distribution.

(iv) If *Z* and *D* are independent, random variables distributed respectively as:

$$Z \sim N_p(0.\Sigma_Z)$$
$$(n-1)D \sim W_p(n - 1.\Sigma_Z)$$

then the quadratic form:

$$T^2 = Z'D^{-1}Z \tag{17}$$

is distributed as:

$$T^2 \sim \frac{(n-1)p}{(n-p)}F_{p.\ n-p} \tag{18}$$

(v) If $Z \sim N_p(0.\Sigma_Z)$ and $(n-1)D \sim W_p(n - 1.\Sigma_Z)$. $(n-1) > p$

where $(n-1)D$ can be decomposed as:

$$(n-1)D = (n-2)D_1 + ZZ'$$

where:

$$(n-2)D_1 \sim W_p(n - 2.\Sigma_Z)$$

and Z is independent of D_1 then the quadratic form:

$$T^2 = Z^{'} D^{-1} Z \tag{19}$$

is distributed as:

$$T^2 \sim (n-1)\beta\ (p.n-p-1)$$

where:

$\beta\ (p.n-p-1)$ is the central Beta distribution.

(vi) If the sample is composed of *k* subgroups of size *n* with subgroup means$\bar{X}_j.\ j = 1.2.3.\dots.k$ and grand mean $\bar{\bar{X}}$. i.e.

$$\bar{X} = \sum_{j=1}^{k} \bar{X}_j / k = \sum_{j=1}^{k} \sum_{i=1}^{n} X_{ij} / kn$$

then

$$\sqrt{\frac{kn}{k-1}} \left(\bar{X}_j - \bar{\bar{X}}\right) \sim N_p\ (0.\Sigma) \tag{20}$$

(vii) If the sample is composed of *K* subgroups of *n* identically distributed multivariate normal observations and if S_j is the sample covariance matrix from the j^{th}subgroup, $j = 1, 2, 3, \dots, K$) then :

$$\Sigma(n-1)S_j \sim W_p\ (K_{n-1}.\Sigma) \tag{21}$$

these distributional properties of $\bar{X}.\ S\ and\ T^2$ are used in the multivariate quality control procedures.
Now, we present two versions of Hotelling T^2 chart:

a) Subgroup Data:

Suppose that *P*-related quality characteristic $X_1.X_2.X_3.\dots.X_p$ are controlled jointly according to the *P*-multivariate normal distribution. The procedure requires computing the sample mean for each of the *P*-quality characteristics from a sample of size *n*.
Let the set of quality characteristic means is represented by the $(p \times 1)$ vector $\bar{X}$ as:

$$\bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_p \end{bmatrix}$$

Then the test statistic plotted on the Chi-square control chart for each sample is:

$$\chi_0^2 = n\left(\underline{\bar{X}} - \underline{\mu}\right)^{'} \Sigma^{-1} \left(\underline{\bar{X}} - \underline{\mu}\right) \tag{22}$$

where: $\underline{\mu}' = (\mu_1.\mu_2.\dots.\mu_p)$ is the $(p \times 1)$ vector of in-control means for each quality characteristic and Σ is covariance matrix.

Now, suppose that *m*-subgroup are available. The sample means and variances are calculated from each subgroup as usual that is:

$$\bar{X}_{jk} = \frac{1}{n}\sum_{i=1}^{n} X_{ijk}$$

$$S_{jk}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_{ijk} - \bar{X}_{jk}\right)^2.$$

$$j = 1.2.\dots.p\ ;k = 1.2.\dots.m \tag{23}$$

where X_{ijk} is the i^{th} observation on the j^{th} quality characteristic in the k^{th}subgroup.

$$S_{jhk} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_{ijk} - \bar{X}_{jk}\right)(X_{ihk} - \bar{X}_{hk}).$$

$$k = 1.2.\dots.n\ .j \neq h \tag{24}$$

represents the covariance between quality characteristic j and quality characteristic h in the k^{th} subgroup.

The statistics $\bar{X}_{jk}.\ S_{jk}^2.\ S_{jhk}$ are the averaged over all m-subgroups to obtain

$$\bar{\bar{X}}_j = \frac{1}{m}\sum_{k=1}^{m} \bar{X}_{jk}\ .\ j = 1.2.\dots.p$$

$$\bar{S}_j^2 = \frac{1}{m}\sum_{k=1}^{m} S_{jk}^2\ .\ j = 1.2.\dots.p$$

and

$$\bar{S}_{jh} = \frac{1}{m}\sum_{k=1}^{m} S_{jhk} \tag{25}$$

where $j \neq h\ and\ \bar{\bar{X}}_j$ are the i^{th} elements of the $(p \times 1)$ sample mean vector $\underline{\bar{\bar{X}}}\ and\ (p \times p)$ average of sample covariance matrices S is formed as:

$$S = \begin{bmatrix} \bar{S}_1^2 \bar{S}_{12} & \dots & \bar{S}_{1p} \\ \bar{S}_2^2 & \dots & \bar{S}_{2p} \\ & & \dots \\ & \bar{S}_p^2 & \end{bmatrix} \tag{26}$$

There are consider the unbiased estimate of $\mu\ and\ \Sigma$ when the process is in control.
If, we replace μ with $\bar{\bar{X}}\ and\ \Sigma$ with S in (22), the test statistic now becomes

$$T^2 = n\left(\bar{X} - \bar{\bar{X}}\right)' S^{-1} (\bar{X} - \bar{\bar{X}}) \tag{16}$$

Alt (1985) has pointed out that there are two distinct phases of control chart using. Phase (I) is the use of the chart for establishing control, that is, testing whether the process was in control when m- subgroups were drawn and the sample statistic $\bar{\bar{X}}$ and S computed. The objective in phase (I) is to obtain an in-control set of observations, so that control limits can be established for phase (II) which is the monitoring of future production.

In the phase (I) the control limits for the T^2-control chart is given by:

$$\left.\begin{aligned} \text{UCL} &= \tfrac{P(m-1)(n-1)}{mn-m-p+1} F_{\alpha.\ P.\ mn-m-p+1} \\ \text{LCL} &= 0 \end{aligned}\right\} \tag{28}$$

In the phase (II) when the chart is used for monitoring future production, the control limits are as follows:

$$\left.\begin{aligned} \text{UCL} &= \tfrac{P(m+1)(n-1)}{mn-m-p+1} F_{\alpha.\ P.\ mn-m-p+1} \\ \text{LCL} &= 0 \end{aligned}\right\} \tag{29}$$

when the parameters μ and Σ are estimated from a large number of subgroups, it is often to use UCL $= \chi_{\alpha.\ p}^2$ as the upper limit in both phases. Retrospective analysis of samples to test for statistical control and establish control limits also occurs in the univariate control chart setting. For the $\bar{X}$-chart, it is well-known that if use $m\ \geq 20\ or\ 25$, samples, the distribution between phase I and phase II limits is usually unnecessary, because the phase I and phase II limits will nearly coincide. However, with multivariate control charts, we must be careful.

Lowry and Montgomery (1995) showed that in many situations a large number of samples would be required before the exact phase II control limits are well approximate by the Chi-square.

b) Individual Observations:

In some situation the subgroup size is naturally n = 1. Suppose that m samples each of size n = 1 are available and that p is the number of quality characteristics observed in each sample. The Hotelling T^2 statistic becomes:

$$T^2 = (X - \bar{X})' S^{-1} (X - \bar{X}) \tag{30}$$

Ryan (1989) defined the phase II control limits for this statistic as:

$$\left.\begin{array}{l}\text{UCL} = \frac{P(m+1)(m-1)}{m(m-p)} F_{\alpha.\ P.\ m-p} \\ \text{LCL} = 0\end{array}\right\} \tag{31}$$

Jackson (1988) suggested that for large m $(m > 100)$ then we can use an approximate control limit, either

$$\text{UCL} = \frac{P(m-1)}{(m-p)} F_{\alpha.\ P.\ m-p} \tag{32}$$

or

$$\text{UCL} = \chi^2_{\alpha.\ p} \tag{33}$$

Equation (33) is only appropriate if the covariance matrix is known.

Lowry and Montgomery (1995) suggested that if p is large-say $p \geq 10$ then at least 250 samples must be taken (m ≥ 250) before Chi-square upper control limit is a reasonable approximation to the correct value.

Tracy, Young and Mason (1992) point out that if $n = 1$, the phase (I) limits should be based on a beta distribution that is, the phase (I) limits defined as:

$$\left.\begin{array}{l}\text{UCL} = \frac{(m-1)^2}{m} \beta_{\alpha.\ \frac{P}{2}.\ \frac{m-p-1}{2}} \\ \text{LCL} = 0\end{array}\right\} \tag{34}$$

3.2.1 The Average Run Length Performance for a Hotelling's T^2

Mason, Tracy and Young (2002) suggested that the average run length (ARL) for a control procedure is defined as: $\text{ARL} = \frac{1}{p}$,

where P represents the probability of being outside the control region. For a process that is in-control, this probability is equal to α, the probability of type I error. The ARL has a number of uses in both univariate and multivariate control procedures. They suggested that it can be used to calculate the number of observations that one would expect to observe, on average, before a false alarm occurs. This given by: $\text{ARL} = \frac{1}{p}$.

Another use of the ARL is to compute the number of observations one would expect to observe before detecting a given shift in the process. The probability of a type II error. The ARL for detecting the shift is given by: $\text{ARL} = \frac{1}{1-\beta}$.

Multivariate control charts using Hotelling's T^2 statistic is popular and easy to use. A major advantage of Hotelling's T^2 statistic is that it can be shown to be the optimal test statistic for detecting a general shift in the process mean vector for an individual multivariate observation. However, the technique has several practical drawbacks. A major drawback is that when the T^2 statistic indicates that a process is out of control, it does not provide information in which variable or set of variables is out of control. Further, it is difficult to distinguish location shifts from scale shifts since the T^2 statistic is sensitive to both types of process changes.

3.2.2 The MYT Decomposition

Mason, Young and Tracy (MYT) extended that the interpretation of signals from a T^2chart to the setting where there is more than process variables MYT decomposition. The MYT decomposition is the primary tool used in this effort, and they examined many interesting properties associated with it. They showed that the decomposition terms contained information on the residuals generated by all possible Linear regressions of one variable on any subset of the other variables. And they add that to being an excellent aid in locating the source of a signal in terms of individual variables or subsets of variables, this property has another major function. It can be used to increase the sensitivity of the T^2 statistic in the area of small process shifts.

Mason, Tracy and Young (1995) presented decomposition procedure. They considered that, the T^2 statistic for a p-dimensional observation vector $\overline{X} = (X_1.X_2.\dots.X_p)$ can be represented as

$$T^2 = \left(X - \overline{X}\right)^{\backslash} S^{-1}\left(X - \overline{X}\right), \tag{35}$$

where $\overline{X}$ and S are the common estimators of the mean vector and covariance matrix obtained from Historical Data Set (HDS), they partitioned the vector $(X - \overline{X})$ as:

$$(X-\overline{X})^{\backslash}=[(X^{(p-1)}-\overline{X}^{(p-1)}).(X_p-\overline{X}_p)]^{\backslash} ,$$

where $X^{(p-1)\backslash}=(X_1.X_2.\dots.X_{p-1})$ represented the $(p-1)$-dimensional variable vector excluding the p^{th}variable X_p and $\overline{X}^{(p-1)}$ represented the corresponding $p-1$ elements of the mean vector. They also partition the matrix S so that.

$$S=\begin{bmatrix} S_{XX} & s_{xX} \\ s_{xX}^{\backslash} & s_{xx} \end{bmatrix}, \tag{36}$$

where S_{XX} is the $(p-1)\times(p-1)$ covariance matrix for the first $(p-1)$ variables, S_p^2 is the variance of X_p, and s_{xX} is a $(p-1)$ −dimensional vector the containing the covariances between X_p and the remaining $(p-1)$ variables.

The T^2statistic in (35) can be partitioned into two independent parts (see Rencher (1993)). These components are given by

$$T^2=T_{p-1}^2+T_{p.1.2.\dots.p-1}^2 . \tag{37}$$

The first term in (37),

$$T_{p-1}^2=(X^{(p-1)}-\overline{X}^{(p-1)})^{\backslash}S_{XX}^{-1}(X^{(p-1)}-\overline{X}^{(p-1)}), \tag{38}$$

uses the first $(p-1)$ variables and is itself a T^2statistic.

Mason, Tracy and Young (1995) proved that the last term in (37) was the $p^{\,th}$ component of the vector X_i adjusted by the estimates of the mean and standard deviation of the conditional distribution X_p given $(X_1.X_2.\dots.X_{p-1})$. It is given by

$$T_{p.1.2.\dots.p-1}^2=\frac{(X_p-\overline{X}_{p.1.2.\dots.p-1})^2}{S_{p.1.2.\dots.p-1}^2}, \tag{39}$$

where $\overline{X}_{p.1.2.\dots.p-1}=\overline{X}_p+B_p^{\backslash}(X^{(p-1)}-\overline{X}^{(p-1)})$, and $B_p^{\backslash}=S_{XX}^{-1}s_{xX}$,is the $(p-1)$-dimensional vector estimate of the coefficients from the regression of X_p on the $(p-1)$ variables $X_1.X_2.\dots.X_{p-1}$. It can be shown that the estimate of the conditional variance is given as $S_{p.1.2.\dots.p-1}^2=S_p^2-s_{xX}^{\backslash}S_{XX}^{-1}s_{xX}$, since the first term of (37) is a T^2statistic, it too can be separated into two orthogonal parts: $T_{p-1}^2=T_{p-2}^2+T_{p-1.1.2.\dots.p-2}^2$. the first term, T_{p-2}^2 , is a T^2statistic, on the first $(p-2)$ components of the X vector, and the second term $T_{p-1.1.2.\dots.p-2}^2$, is the square of X_{p-1} adjusted by the estimates of the standard deviation of the conditional distribution of X_{p-1} given $(X_1.X_2.\dots.X_{p-2})$. They proposed one from of MYT decompositions of a T^2statistic. It is given by,

$$\begin{aligned} T^2 &= T_1^2+T_{2.1}^2+T_{3.1.2}^2+\ \dots+T_{p.1.2.\dots.p-1}^2 \\ &= T_1^2+\textstyle\sum_{j=1}^{p-1}T_{j+1.1.\dots.j}^2 . \end{aligned} \tag{40}$$

The T_1^2 term in (40) is the square of the univariate for the first variable of the vector X and is given as

$$T_1^2=\frac{(X_1-\overline{X}_1)^2}{S_1^2}. \tag{41}$$

This term is not a conditional term, as its value does not depend on a conditional distribution. In contrast, all other terms of the expansion in (40) are conditional terms, since they represent the value of a variable by the mean and standard deviation from the appropriate conditional distribution.

Computing the Decomposition Terms:

They considered that the first $(p-1)$ terms of (40) correspond to the T^2 value of the sub vector $X_{p-1}^{\backslash}=(X_1.X_2.\dots.X_{p-1})$; i.e., $T_{(X_1.X_2.\dots.X_{p-1})}^2=T_1^2+T_{2.1}^2+T_{3.1.2}^2+\ \dots+T_{p-1.1.2.\dots.p-2}^2$,

similarly, the first $(p-2)$ terms of this expansion correspond to the sub vector $X_{p-2}^{\backslash}=(X_1.X_2.\dots.X_{p-2})$; i.e.,

$$T_{(X_1.X_2.\dots.X_{p-2})}^2=T_1^2+T_{2.1}^2+T_{3.1.2}^2+\ \dots+T_{p-2.1.2.\dots.p-3}^2 .$$

continuing in this fashion, they compute the T^2 values for all sub vectors of the original vector X. The last sub vector, consisting of the first component $X_{(1)}=(X_1)$ is used to compute the unconditional T^2 term given in (41) ; i.e.,$T_{X_1}^2=T_1^2$.

All the T^2values, $T_{(X_1.X_2.\dots.X_p)}^2$. $T_{(X_1.X_2.\dots.X_{p-1})}^2$. $T_{(X_1)}^2$, are computed using the general formula

$$T_{(X_1.X_2.\dots.X_j)}^2=\left(X^{(j)}-\overline{X}^{(j)}\right)^{\backslash}S_{jj}^{-1}(X^{(j)}-\overline{X}^{(j)}), \tag{42}$$

where $X^{(j)}$ represents the appropriate sub vector. $\overline{X}^{(j)}$ is the corresponding sub vector mean and S_{jj} denotes the corresponding covariance sub matrix obtained from the overall S matrix given in (36) by deleting all unused rows and columns. The terms of the MYT decomposition can be computed as follows

$$T_{p.1.2.\dots.p-1}^2=T_{(X_1.X_2.\dots.X_p)}^2-T_{(X_1.X_2.\dots.X_{p-1})}^2$$

$$T^2_{p-1.1.2.....p-2} = T^2_{(X_1.X_2.....X_{p-1})} - T^2_{(X_1.X_2.....X_{p-2})}$$
$$\dots\dots$$
$$\dots\dots$$
$$\dots\dots$$
$$T^2_{2.1} = T^2_{(X_1.X_2)} - T^2_1$$
$$T^2_1 = \frac{(X_1-\overline{X}_1)^2}{S^2_1}. \quad (43)$$

Properties of the MYT Decomposition:

Many properties are associated with the MYT decomposition. Consider they dimensional vector defined as $X^\backslash = (X_1.X_2.\dots.X_p)$ they interchange the first two components to form another vector $(X_2.X_1.\dots.X_p)$ so that the only difference between the two vectors is the two vectors such that the first two components have been permuted the T^2value of the two vectors is the same; i.e. $T^2_{(X_1.X_2.....X_p)} = T^2_{(X_2.X_1.....X_p)}$.

This occurs because T^2 values cannot be changed by permuting the components of the observation vector. This invariance property of permuting the T^2 components that each ordering of an observation vector will produce the same overall T^2 value. Since there are $p! = (p)(p-1)(p-2)\dots(2)(1)$ permutations of the components of the vector $(X_1.X_2.\dots.X_p)$, this implies that we can partition a T^2 value in $p!$ different ways. To illustrate his result, suppose $p = 3$. There are $3! = (3)(2)(1) = 6$ decompositions of the T^2 value for an individual observation vector. These are listed below:

$$\begin{aligned} T^2 &= T^2_1 + T^2_{2.1} + T^2_{3.1.2} \\ &= T^2_1 + T^2_{3.1} + T^2_{2.1.3} \\ &= T^2_2 + T^2_{3.2} + T^2_{1.2.3} \\ &= T^2_2 + T^2_{1.2} + T^2_{3.1.2} \\ &= T^2_3 + T^2_{1.3} + T^2_{2.1.3} \\ &= T^2_3 + T^2_{2.3} + T^2_{1.2.3}. \end{aligned} \quad (44)$$

Each row of (44) corresponds to a different permutation of the components of the observation vector. For example, the first row corresponds to the vector written in its original form as $(X_1.X_2.X_3)$, whereas the last row represents $(X_3.X_2.X_1)$. Note that all six possible permutations of the original vector components are included.

The importance of this result is that it allows one to examine the T^2 statistic from many different perspectives. The p terms in any particular decomposition are independent of one another, although the terms across the decompositions are not necessarily independent. With p partition and $p!$ partitions, there are $p \times p!$ possible terms to evaluate in a total MYT decomposition of a particular partition, as certain terms occur more than once. In general, there are $p \times 2^{(p-1)}$ distinct terms among the possible decompositions. These unique terms are the ones that need to be examined for possible contribution to a T^2 signal, when p is large. Computing all these terms can be cumbersome.

They considered the MYT decomposition given in (40) and suppose T^2_1 dominates the overall value of the T^2 statistic. This indicates that the observation on the variable X_1 is contributing to the signal. However, to determine if the remaining variables in this observation contribute to the signal, we must examine the T^2 value associated with the sub vector $(X_2.X_3.\dots.X_p)$ which excludes the X_1 component. Small values of the T^2statistic for this sub vector imply that no signal is present. They also indicate that one need not examine any term of the total decomposition involving these $(p-2)$ variables.

They considered another important property ofT^2statistic is the fact that the $p(2^{(p-1)} - 1)$ unique conditional terms of a MYT decomposition contain the residuals from all possible linear regressions of each variable on all subsets of the other variables. This property of the T^2 statistic provides a procedure for increasing the sensitivity of the T^2 sensitivity of the T^2 statistic to process shifts.

Locating Signaling Variables:

They seek to relate a T^2 signal and its interpretation to the components of the MYT decomposition. They consider signaling observation vector $X^\backslash = (X_1.X_2.\dots.X_p)$ such that$T^2_{(X_1.X_2.....X_p)} > \mathrm{UCL}$.

They proposed two methods one for locating the variables contributing to the signal is to develop a forward iterative Scheme. This was accomplished by finding the subset of variables that do not contribute to signal from (37) and (39) such that a T^2 statistic can be constructed on any subset of the variables $X_1.X_2.\dots.X_p$. Construct the T^2statistic for each individual variable X_j. $j = 1.2.\dots.p$, so that

$$T^2_j = \frac{(X_j - \overline{X}_j)^2}{S^2_j}.$$

where $\overline{X}_j$ and S^2_j are the corresponding mean and variance estimates as determined from the HDS. Compare these individual T^2_j values to their UCL where

$$\text{UCL}_{(X_j)} = \left\{\frac{p(n+1)(n-1)}{n(n-p)}\right\} F_{(\alpha.p.n-p)}$$
$$= \left\{\frac{n+1}{n}\right\} F_{(\alpha.1.n-1)}, \tag{45}$$

is computed for an appropriate α level and for a value of $p = 1$. Exclude from the original set of variables all X_j for which

$$T_j^2 > \text{UCL}_{(X_j)}.$$

since observations on this subset of variables are definitely contributing to the signal.

From the set of variables not contributing to the signal. Compute the T^2 statistic for all possible pairs of variables. For example, for all $(X_i.X_j)$ with $i \neq j$ compute $T^2_{(X_i.X_j)}$,and compare these values to the upper control limit,

$$\text{UCL}_{(X_i.X_j)} = \left\{\frac{2(n+1)(n-1)}{n(n-2)}\right\} F_{(\alpha.2.n-2)}.$$

Exclude from this group all pairs of variable for which $T^2_{(X_i.X_j)} > \text{UCL}_{(X_i.X_j)}$.

The excluded pairs of variables in addition to exceeded single variable comprise the group of variables contributing to the overall signal continue to iterate in this fashion so as to exclude from the remaining group all variables of signaling groups of three variables four variables etc. The procedure produces a set of variables that contribute to the signal. And another method of locating the vector components contributing to a sing al is to examine the individual terms of the MYT decomposition of a signaling observation vector and to determine which are large in value. This method was accomplished by comparing each term to its corresponding critical value.

Mason, Tracy, and Young (1995) proposed that the distribution governing the components of the MYT decomposition for the situation where there are no signals is F distribution. For the case of p variables, these are given by

$$T_j^2 \sim \left\{\frac{n+1}{n}\right\} F_{(1.n-1)}, \tag{46}$$

for unconditional terms, and by

$$T^2_{j.1.2.....j-1} \sim \left\{\frac{(n+1)(n-1)}{n(n-k-1)}\right\} F_{(1.n-k-1)}, \tag{47}$$

for conditional terms, where k equals the number of conditioned variables. For $k = 0$, the distribution in (47) reduces to the distribution in (46). Using these distributions, critical values (CV_S), for a specified α level and HDS sample of size n for both conditional and unconditional terms are obtained as follows:

unconditional terms: $\text{CV} = \left\{\frac{n+1}{n}\right\} F_{(\alpha.1.n-1)}$,

conditional terms: $\text{CV} = \left\{\frac{(n+1)(n-1)}{n(n-k-1)}\right\} F_{(\alpha.1.n-k)}$. (48)

They add that we can compare each individual term of the decomposition to its critical value and make the appropriate decision.

Interpretation of a Signal on a T^2 Component:

They Mason, Tracy, and Young (2002) considered one of the p possible unconditional, terms resulting from the decomposition of the T^2statistic associated with a signaling observation. As stated earlier, the term$T_j^2 = \frac{(X_j-\overline{X}_j)^2}{S_j^2}$, $j = 1.2.\dots.p$ is square of a univariate t statistic for the observed value of the j^{th} variable of an observation vector X. For control to be maintained, this component must be less than its critical value i.e.,$T_j^2 < \left\{\frac{n+1}{n}\right\} F_{(\alpha.1.n-1)}$,

since$t_{(\frac{\alpha}{2}.\ n-1)} = \sqrt{F_{(\alpha.1.n-1)}}$ they re-expressed this condition as T_j being in the following interval:

$$-\sqrt{\left(\frac{n+1}{n}\right)} t_{(\frac{\alpha}{2}.\ n-1)} < T_j < \sqrt{\left(\frac{n+1}{n}\right)} t_{(\frac{\alpha}{2}.\ n-1)}, \tag{49}$$

or as

$$\overline{X} - \sqrt{\left(\frac{n+1}{n}\right)} t_{(\frac{\alpha}{2}.\ n-1)} < X_j < \overline{X} + \sqrt{\left(\frac{n+1}{n}\right)} t_{(\frac{\alpha}{2}.\ n-1)}, \tag{50}$$

where$t_{(\frac{\alpha}{2}.\ n-1)}$ is, the appropriate value from a t-distribution with $n-1$ degrees of freedom. This is equivalent to using a univariate Shewhart control chart for the j^{th} variable. And they considered the form of a general conditional term given as

$$T^2_{j.1.2.....j-1} = \frac{(X_j - \overline{X}_{j.1.2.....j-1})^2}{S^2_{j.1.2.....j-1}}, \tag{51}$$

if the value in (51) is to be less than its control limit,

$$T^2_{j.1.2.....j-1} < \left\{\frac{(n+1)(n-1)}{n(n-k-1)}\right\} F_{(\alpha.1.n-k-1)}.$$

Its numerator must be small, as the denominator of these terms is fixed by the historical data this implies that component X_j from the observation vector$X^{\backslash} = (X_1.X_2.....X_j.....X_p)$ is contained in the conditional distribution of X_j given $X_1.X_2.....X_{j-1}$ and falls in the elliptical control region.

A signal occurs on the term in (51) when X_j is not contained in conditional distribution of X_j given $X_1.X_2.....X_j$ i.e. when $T^2_{j.1.2.....j-1} > \left\{\frac{(n+1)(n-1)}{n(n-k-1)}\right\} F_{(\alpha.1.n-k-1)}$.

Regression perspective:

Mason, Tracy and young (1995,1999) proposed that, in general, $T^2_{j.1.2.....j-1}$ is a standardized observation on the j^{th} variable adjusted by the estimated of the mean and variance form the conditional distribution associated with $(X_j | X_1.X_2.....X_{j-1})$. The general from of this term was given in (51). They considered the estimated mean of X_j adjusted for $X_1.X_2.....X_{j-1}$, and estimated this mean by using the prediction equation.

$$\overline{X}_{j.1.2.....j-1} = \overline{X}_j + B^{\backslash}{}_j \left(X^{(j-1)} - \overline{X}^{(j-1)}\right), \tag{52}$$

where $\overline{X}_j$ is the sample mean of X_j obtained from the historical data. The sub vector$X^{(j-1)}$ is composed of the observation on $(X_1.X_2.....X_{j-1})$ and $\overline{X}^{(j-1)}$ is the corresponding estimated mean vector, S_{jj}, is the covariance matrix of the first j components of the vector X. To obtain S_{jj} partition S as follows:

$$S = \begin{bmatrix} S_{jj} & S_{j(p-j)} \\ S^{\backslash}_{j(p-j)} & S_{(p-j)(p-j)} \end{bmatrix}$$

Further, partition the matrix S_{jj} as

$$S_{jj} = \begin{bmatrix} S_{(j-1)(j-1)} & S_{j(j-1)} \\ S^{\backslash}_{j(j-1)} & S^2_j \end{bmatrix}$$

Then

$$B_j = S^{-1}_{(j-1)(j-1)} S_{j(j-1)}$$

since the left-hand side of (52) contains$\overline{X}_{j.1.2.....j-1}$, which is the predicted value of X_j, the numerator of (51) is a regression residual represented by

$$r_{j.1.2.....j-1} = (X_j - \overline{X}_{j.1.2.....j-1}),$$

rewriting the conditional variance as

$$S^2_{j.1.2.....j-1} = S^2_j(1 - R^2_{j.1.2.....j-1})$$

(See, e.g, Rencher (1993)) and substituting $r_{j.1.2.....j-1}$for$(X_j - \overline{X}_{j.1.2.....j-1})$, they expressed

$$T^2_{j.1.2.....j-1} = \frac{(r_{j.1.2.....j-1})^2}{S^2_j(1-R^2_{j.1.2.....j-1})} \tag{53}$$

The above results indicate a T^2 signal may occur if something goes astray with the relationships between subsets of various variables. This situation can be determined by examination of the conditional T^2 terms. A signaling value indicates that a contradiction with historical relationship between the variables has occurred either (I) due to a standardized component value that is significantly larger or smaller than that predicted by a subset of the remaining variables or (II) due to a standardized component value that is marginally smaller or larger than that predicted by a subset of the remaining variables when there is a very severe collinearity (i.e., a large R^2 value) among the variables. Thus, a signal results when an observation on a particular variable or set of variables, is out of control and/or when observations on a set of variables are counter to the relationship established by the historical data.

Improving the Sensitivity of the T^2Statistic:

Mason, Tracy and young (1999 - 2002) used the decomposition for improving the sensitivity of the T^2 in signal detection. They showed that the T^2 statistic to be a function of all possible regressions existing among a set of process variables. Furthermore, they showed that the residuals of the estimated regression models are contained in the conditional terms of the MYT decomposition. Large residuals produce large T^2 components for the conditional terms and are interpreted as indicators of counter relationships among the variables. However, a large residual also could imply an incorrectly specified model. This result suggests that it may be possible to improve the performance of the T^2 statistic by more carefully describing the functional relationships existing among the process variables. Minimizing the effects of model misspecification on the signaling ability of the T^2 should improve its performance in detecting abrupt process shifts. They showed when compared to other multivariate control procedures, the T^2 Lacks the sensitivity of detecting small process shifts. They showed that this problem can be overcome by monitoring the error residuals of the regressions contained in the conditional terms of the MYT decomposition of T^2 statistic. Furthermore, they showed that such monitoring can be helpful in certain types of on-line experimentation within a processing unit.

They proposed an alternative form of condition terms, they considered the conditional term of the MYT decomposition in (51) This is the squares of the j^{th} variable of the observation vector which adjusted by the estimates of the mean and variance of the conditional distribution of X_j given $X_1.X_2.\dots.X_{j-1}$. They showed that (51) could be written as

$$T^2_{j.1.2.\dots.j-1} = \left(\frac{r_{j.1.2.\dots.j-1}}{s_{j.1.2.\dots.j-1}}\right)^2 \tag{54}$$

this was achieved by noting that $\overline{X}_{j.1.2.\dots.j-1}$can be obtained from the regression of $X_j$on$X_1.X_2.\dots.X_{j-1}$: i.e.,

$$\overline{X}_{j.1.2.\dots.j-1} = b_0 + b_1X_1 + b_2X_2 + \cdots + b_{j-1}X_{j-1} \tag{55}$$

where b_j are the estimated regression coefficients. Since $\overline{X}_{j.1.2.\dots.j-1}$ is the predicted value of X_j. The numerator of (51) is the raw regression residual,

$$r_{j.1.2.\dots.j-1} = (X_j - \overline{X}_{j.1.2.\dots.j-1}) \tag{56}$$

given in (54) .

Another form of the conditional term in (51) is obtained by substituting the following quantity for the conditional variance contained in (54) i.e., by substituting

$$S^2_{j.1.2.\dots.j-1} = S^2_j(1 - R^2_{j.1.2.\dots.j-1}) ,$$

where $R^2_{j.1.2.\dots.j-1}$is the squared multiple correlation between X_j and $X_1.X_2.\dots.X_{j-1}$ this yields:

$$T^2_{j.1.2.\dots.j-1} = \frac{r^2_{j.1.2.\dots.j-1}}{S^2_j(1-R^2_{j.1.2.\dots.j-1})} \tag{57}$$

Much information is contained in the conditional terms of the MYT decomposition. Since these terms are, in fact, squared residuals from regression equations they can be helpful in improving the sensitivity of the T^2 statistic in detecting both abrupt process changes and gradual shifts in the process.

They considered the better the fit of a model, the more sensitive the T^2 control procedure will be to departures from the model. This suggests that more effort should be taken in phase 1 operations, during construction of the historical database, to insure proper functional forms are chosen for the process variables and useful models are created.

Principal Components Procedure:

Jakson (1980, 1991) recommended to use the principal components procedure to aid in the interpretation of an out-of-control signal. The principal component analysis (PCA) technique decomposes the T^2 statistic into a sum of p independent principal components, which are linear combinations of the original variables, and using these components to help for solving this identification problem. The PCA used to reduce to dimensionality of a data set which consists of a large number of interrelated variables. While retaining as much as possible of the variation present in the data set. This goal is achieved by transforming the original variables to a new set of uncorrelated variables, which are called the principal components. This transformation is a principal axis rotation of the variance and covariance matrix of the data set, and the elements of the characteristic vectors or the eigenvectors of the covariance matrix are direction cosines of the new axes related to the old.

The transformed new uncorrected variable or the principal components are normally numbered in descending order according to the amount at the variation. The use of the method of PCA in the field of multivariate quality control was first introduced by Jackson and Morris (1957). They identified a large number (p) of correlated variables that account for the quality of the process.

They notice that the use of Hotelling's T^2 may involve computational problems since the determinant of the variance and covariance matrix is near zero. The solution is to transform the original p variables to lesser k principal components.

Jackson (1980) proposed using PCA to interpretation an out-of- control single by decomposing T^2 into independent component. Jackson proposed that the starting point of the statistical application of the method of principal components is the sample covariance matrix *S*. for a *p*-variate problem in (13).

If the covariances are not equal to zero, it indicates a relationship existing between those two variables, the strength of that relationship (if it is linear) being represented by the correlation $r_{ij} = \frac{S_{ij}}{(S_i S_j)}$.

A principal axis transformation will transform p correlated variables $X_1.X_2.\dots.X_p$ into p new uncorrelated variables $Z_1.Z_2.\dots.X_p$ the coordinate axes of these new variables being described by the vectors u_i which make up the matrix *U* of direction cosines used in the following transformation:

$$Z = U^{\setminus}\left(X - \overline{X}\right) \tag{58}$$

the transformed variables are called the principal components of X. The *j*th principal component would be

$$z_i = u_i^{\setminus}\left(X - \overline{X}\right) \tag{59}$$

If one wishes to transform a set of variables X by a linear transformation $Z = U^{\setminus}\left(X - \overline{X}\right)$, whether U is orthogonal or not, the covariance matrix of the new variables S_z can be determined directly from the covariance matrix of the original variables S by the relationship:

$$S_z = U^{\setminus}\, S\, U \tag{60}$$

However, the fact that *U* is orthonormal is not a sufficient condition for the new variables to be independent. Only a transformation such as the principal axis transformation will produce the diagonal element S_z of the matrix *L* where

$$L = \begin{bmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_p \end{bmatrix}$$

The fact that S_z is diagonal elements of L means that principal components are uncorrected. He added that we can also determine the correlation of each principal component with each of the original variables; this is useful for diagnostic purposes. The correlation of the *i*th principal component S_z and *j*th original variable X_j can be determined as

$$r_{z_i X_j} = \frac{u_{ij}\sqrt{L_i}}{s_j} \tag{61}$$

Another interesting property of principal components is the fact that the equation (58) can be inverted to

$$X = \overline{X} + U_z \tag{62}$$

by virtue of the fact that U is orthonormal so that$U^{-1} = U^{\setminus}$. This means that if we know the values of the principal components, we can determine what the original data were.

Principal Component Form of T^2:

Jackson (1991) proposed that the T^2 can be expressed as a function of the principal components of the estimated covariance matrix. He gives an alternative form of the T^2 statistic as:

$$T^2 = \left(X - \overline{X}\right)^{\setminus} S^{-1}\left(X - \overline{X}\right) = \sum_{i=1}^{p} \frac{z_i^2}{\lambda_i} \tag{63}$$

where$\lambda_1 > \lambda_2 > \cdots > \lambda_p$ are the eigenvalues of the estimated covariance matrix *S* and the z_i, $i = 1.2.\dots.p$, are the corresponding principal components.

Each of these component is obtained by multiplying the vector quantity $\left(X - \overline{X}\right)$ by the transpose of the normalized eigenvector U_i of *S* corresponding toλ_i: i.e.,$z_i = U_i^{\setminus}\left(X - \overline{X}\right)$

Each z_i is a scalar quantity and the T^2 statistic is expressed in terms of these values.

The representation in (63) is derived from the fact that the estimated covariance matrix *S* is a positive definite symmetric matrix. Thus, its singular value decomposition is given as$S = UAU^{\setminus}$, where *U* is a *p*×*p* orthogonal matrix whose columns are the normalized eigenvectors U_i of S, and A is a diagonal matrix whose elements are the corresponding eigenvalues.

$$U = (U_1, U_2, ..., U_p) \quad \text{and} \quad A = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & \lambda_p \end{bmatrix}$$

$$S^{-1} = UA^{-1}U^{\backslash} \tag{64}$$

Substituting this quantity into the T^2 statistic of (63) we have

$$T^2 = (X_i - \overline{X})^{\backslash} UA^{-1}U^{\backslash}(X_i - \overline{X})$$

$$= Z^{\backslash} A^{-1} Z = \sum_{i=1}^{p} \frac{Z_i^2}{\lambda_i} \tag{65}$$

where $z_i = U_i^{\backslash}(X - \overline{X})$ and $Z^{\backslash} = z_1.z_2.\dots.z_p$.

A Hotelling's T^2 statistic for a single observation also can be written as

$$T^2 = (X - \overline{X})^{\backslash} S^{-1}(X - \overline{X}) = Y^{\backslash} R^{-1}\, Y,$$

where R is the estimated correlation matrix and Y is the standardized observation vector of x, i.e.,

$$R = \begin{bmatrix} 1 & r_{12} & r_{13} & ... & r_{1p} \\ r_{21} & 1 & r_{23} & ... & r_{2p} \\ ... & r_{32} & 1 & ... & r_{3p} \\ ... & ... & ... & ... & ... \\ r_{p_1} & ... & ... & r_{p(p-1)} & 1 \end{bmatrix}$$

where $r_{ij} = corr(X_1.X_j)$and

$$\left[\frac{(X_1 - \overline{X})}{S_1}.\frac{(X_2 - \overline{X})}{S_2}.\dots.\frac{(X_p - \overline{X})}{S_p}\right] = [y_1.y_2.\dots.y_p]$$

The matrix R (obtained from S) is a positive definite symmetric matrix and can be represented in terms of its eigenvalues and eigenvectors. Using a transformation similar to (64). the above T^2 can be written as

$$T^2 = \sum_{i=1}^{p} \frac{W_i^2}{\gamma_i} \tag{66}$$

where $\gamma_1 > \gamma_2 > \dots > \gamma_p$ are the eigenvalues of the correlation matrix R, and $w_1.w_2.\dots.w_p$ are the corresponding principal components of the matrix R. w_i canbe determined by the following transformation$w_i = v_i^{\backslash}(X - \overline{X})$, i=1,2, ..., p , where the$v_{1.}v_{2.}\dots.v_p$ are the corresponding normalized eigenvectors of R Equation (66) is not to be confused with (63). The first equation is written in terms of the eigenvalues and eigenvectors of the covariance matrix, and the second is in terms of the eigenvalues and eigenvectors of the estimated correlation matrix. These are two different forms of the same Hotelling's T^2 as the mathematical transformations are not equivalent.

The principal component representation of the T^2 plays a number of roles in multivariate statistical process control (SPC). The control region can be defined by the UCL. The observations contained in the T^2 values less than the UCL. i.e., for each X_i,

$$T_i^2 < UCL$$

Thus, by (66)

$$T^2 = \frac{w_1^2}{\gamma_1} + \frac{w_2^2}{\gamma_2} + \dots\dots + \frac{w_p^2}{\gamma_p} < \text{UCL}$$

The control region is defined by the equality.

$$\frac{w_1^2}{\gamma_1}+\frac{w_2^2}{\gamma_2}+........+\frac{w_p^2}{\gamma_p}=\text{UCL},$$

Which is the equation of a hyper ellipsoid in a *p*-dimensional space provided the eigenvalues $\gamma_1 > \gamma_2 > ... > \gamma_p$ are all positive. The fact that the estimated correlation matrix *R* is a positive definite matrix guarantees that all the γ_i's are positive.

Note that in special case of the principal component space of the estimated correlation matrix. T^2can be reduced to

$$T^2=\frac{w_1^2}{\gamma_1}+\frac{w_2^2}{\gamma_2}=\text{UCL} \qquad (67)$$

which gives the equation of the control ellipse. The length of the major axis of the ellipse in (67) is given by γ_1 and the length of the minor axis is given by γ_2. The axes of this space are the principal components, w_1 and w_2. The absence of a product term in this representation indicates the independency between w_1 and w_2.This is a characteristic of principal components, since they are transformed to be independent.

Assuming that the estimated correlation r is positive it can be shown that $\gamma_1=(1+r)$ and $\gamma_2=(1-r)$. For negative correlations, the γ_i values are reversed. One can also show that the principal components can be expressed as $w_1=(y_1+y_2)\sqrt{2}$, $w_2=(y_2-y_1)\sqrt{2}$. From these equations, one can obtain the principal components as functions of the original variables.

IV. THE APPLICATION

Delta Fertilizers and Chemical Industries is considered one of the leading companies in the field of fertilizers production in Egypt. About 4500 employees are working for it, on the various managerial levels. Urea production is one of the major products of the company. The production of urea occurs through three stages, summarized as follows:

A. High pressure stage

In this stage, urea is produced through two reactions; the first reaction occurs by condensation of Ammonia Gas and Carbon dioxide under high pressure and temperature for the sake of the production of intermediate material, known as Carbamate. The second reaction happens by separating the water from the Carbamate in order to a chive urea. In this, stage the condensation of urea approximately 56%.

It contains 16 variables, these are:

1	X1	E-201Outlet Temperature
2	X2	Outlet cold NH3 from E- 201
3	X3	CO_2 to Train
4	X4	CO_2 pressure to synthesis
5	X5	CO_2 after E-22
6	X6	R-201
7	X7	Temperature in reactor R-201
8	X8	Temperature in reactor R-201
9	X9	Temperature in reactor R-201
10	X10	Temperature in reactor R-201
11	X11	Stripper level
12	X12	Liquid leaving the Stripper
13	X13	Stream from E-204 to j-201
14	X14	Conditioned water to scrubber E-204
15	X15	Conditioned water from scrubber E-204
16	X16	Stream from j-203

Table analysis of laboratory in this stage:

1	t1.1	NH_3	Reactor outlet

2	t1.2	CO_2	Reactor outlet
3	t1.3	UR	Reactor outlet
4	t1.4	B_1	Reactor outlet
5	t1.5	H_2O	Reactor outlet
6	t2.1	NH3	Stripper outlet
7	t2.2	CO_2	Stripper outlet
8	t2.3	UR	Stripper outlet
9	t2.4	B_1	Stripper outlet
10	t2.5	H_2O	Stripper outlet

B. Low pressure stage

In this stage, the condensation of urea liquid rises from 56% to 71%. This happens through the decomposition of the remaining Carbamate and the elimination of water under low pressure.

It contains seven variables, these are:

1	y1	Urea solution from stripper E-202
2	y2	Steam to E-205
3	y3	Urea carbonate solution from stripper T-201 to E-205
4	y4	Gas leaving T-201
5	y5	Level in TK-201
6	y6	P-203
7	y7	Urea solution in TK-201

Table analysis of laboratory in this stage:

1	t3.1	NH_3	D 202 Outlet
2	t3.2	CO_2	D 202 Outlet
3	t3.3	UR	D 202 Outlet
4	t3.4	B_1	D 202 Outlet
5	t3.5	H_2O	D 202 Outlet
6	t4.1	NH_3	In TK 201
7	t4.2	CO_2	In TK 201
8	t4.3	UR	In TK 201
9	t4.4	B_1	In TK 201
10	t4.5	H_2O	In TK 201
11	t5.1	NH_3	In PI 302
12	t5.2	CO_2	In PI 302
13	t5.3	UR	In PI 302

C. Evaporation and prilling stage

This stage occurs by two stages:

(i) Evaporation stage

In this stage, the condensation of urea rises from 71% to 98.7% approximately and the urea liquid trams forms to urea melt. This happens under high pressure and temperature.

(ii) Prilling stage

In this stage, the urea melt is through formed into prilling in the prilling tower.

It contains four variables, these are:

1	Z1	Urea solution from D-204 to E-209
2	Z2	D- 205 Vacuum
3	Z3	Urea to prilling tower X-202

4	Z4	E- 211 Vacuum

Table analysis of laboratory in this stage:

1	t6.1	B1
2	t6.2	H_2O
3	t6.3	Pills > 3.35
4	t6.4	Pills 3.35: 2.4
5	t6.5	Pills 2.4: 1.4
6	t6.6	Pills 1.4: 1.0
7	t6.7	Pills < 1.0
8	t6.8	UR

Data Description:

For the application of multivariate quality control, chart data originate from urea production process, which consists of the three stages and the analysis of laboratory, which discussed above.
The number of the sample is 732 observations taken per hour.
The advantages of this sample that, it has several variables and several stages of the production. This advantage of the production is the basic reason for choosing this production to allow us to study the multivariate quality control charts.

In this application, we shall introduce the most common using technique of multivariate quality control charts; MEWMA control chart &Hotelling's $\boldsymbol{T^2}$ control chart.

4.1 MEWMA Control Chart

A MEWMA control chart consists of:

- Plotted points, each of which represents the multivariate statistic for each observation.
- Upper control limits (red), which provide a visual means for assessing whether the process is in-control.

MINITAB marks points outside of the control limits with a red symbol
We select combination of r and ARL for plotting several MEWMA charts for each stage of the production study.
Note that the default value of *r* = 0.3 and the default value of ARL= 200.

4.1.1 MEWMA control chart of X1, ..., X16 and t1.1, …, t2.5

Test Results for MEWMA Chart of X1, ..., X16 and t1.1, …, t2.5
TEST. One point beyond control limits.
Test Failed at points:

1	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
64	65	66	67	70	71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95
96	97	98	99	100	101	102	103	104	105	106	107	108	109	110
111	112	113	177	178	179	180	181	182	183	184	185	186	187	188
189	190	191	192	193	194	195	196	197	198	199	200	201	202	208
209	211	212	213	216	218	219	220	222	225	226	227	228	229	230
231	232	234	235	237	238	240	241	243	244	245	246	247	248	249
250	252	253	254	255	258	259	261	262	263	264	265	267	268	269
271	272	274	275	276	277	278	279	280	281	282	283	284	286	287
288	292	293	294	295	296	297	298	299	300	301	302	303	304	305
306	307	308	309	310	311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330	331	332	333	334	335
336	337	338	339	340	341	342	343	344	345	346	347	348	359	360
366	367	368	383	391	392	400	401	402	403	404	405	406	407	408
409	410	411	412	413	414	415	416	417	418	419	420	421	422	423
424	429	430	430	432	433	434	435	436	437	438	439	440	441	442
443	444	445	446	447	448	449	450	451	452	453	454	455	456	457
458	459	460	461	462	463	464	465	466	467	468	469	470	471	472
473	474	475	476	477	478	479	480	481	482	483	484	485	486	487

488	489	496	498	499	500	502	504	506	508	510	512	514	515	516
517	518	520	524	525	526	527	528	529	530	531	532	533	534	535
536	537	538	539	540	541	542	543	544	545	546	547	548	549	550
551	552	553	554	555	556	557	558	559	560	561	562	563	564	565
566	567	568	569	570	571	572	573	636	637	647	648	657	658	659
660	661	663	664	665	666	672	674	675	676	677	678	679	680	681
682	683	684	685	686	687	688	689	690	691	692	693	694	695	696
697	698	699	700	701	702	703	704	705	706	707	708	709	710	711
712	713	714	715	716	717	718	719	720	721	722	723	724	725	726
727	728	729	730	731	732									

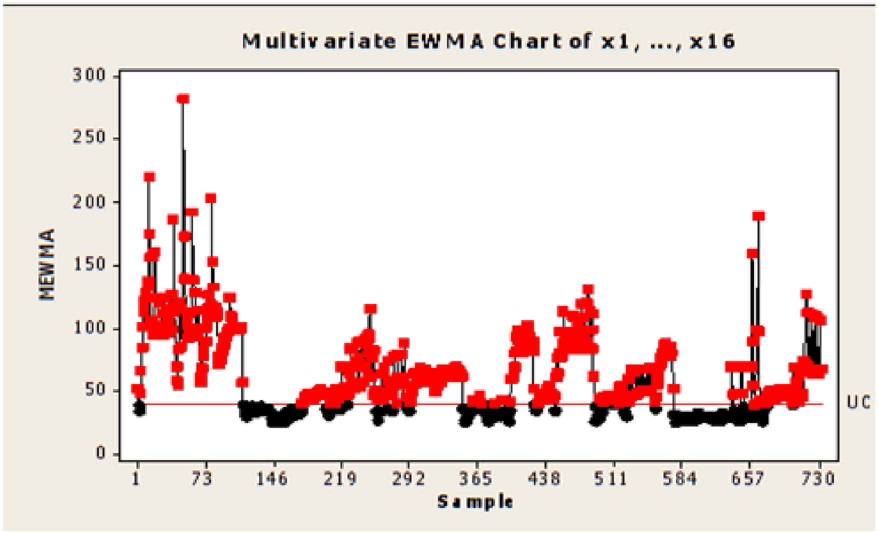

Figure1. MEWMA chart of X1, ..., X16 and t1.1, …, t2.5.

The MEWMA chart of X1, ..., X16 and t1.1, …, t2.5 can be summarized as follows:

- The upper control limit is 40.5. Therefore, we expect the MEWMA statistics to fall below 40.5.
- Test results indicate that 491 points beyond the control limits.
- Test results indicate that the process is in- control for 241 points and out –of control for 491 points. Then the out-of-control rate 67.08% and the in-control rate 32.92%.

4.1.2 MEWMA control chart of y1, …, y7 and t3.1, …, t4.5

Test Results for MEWMA Chart of y1, …, y7 and t3.1, …, t4.5

TEST. One point beyond control limits.

Test Failed at points:

6	7	8	9	10	11	12	13	14	15	16	17	19	21	22
24	28	29	30	31	32	33	35	37	51	53	55	56	57	58
59	61	62	63	65	67	75	77	82	87	89	91	92	93	94
95	96	99	107	108	110	111	114	115	116	130	131	132	148	149
150	151	152	153	154	155	156	157	158	159	160	161	162	163	164
165	166	167	168	169	170	171	172	173	174	181	182	183	184	185
186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
201	202	203	206	207	208	209	229	230	247	250	251	252	253	254
256	258	263	264	265	267	268	269	271	272	274	275	276	278	279
280	281	282	283	284	285	286	287	288	289	292	293	294	295	296
297	298	299	300	301	302	303	304	305	306	307	308	309	310	311
312	313	314	315	316	317	318	352	353	354	355	356	357	358	359
360	361	362	363	364	365	366	367	368	369	370	371	372	373	374
386	454	457	458	460	463	464	465	466	467	468	469	470	471	472
473	474	475	476	477	478	479	480	481	482	483	484	485	486	487
488	489	517	551	552	553	554	555	703	704	705	706	714	715	716
717	718	719	720	721	722	723	724	725	726	727	728	729	730	731

732

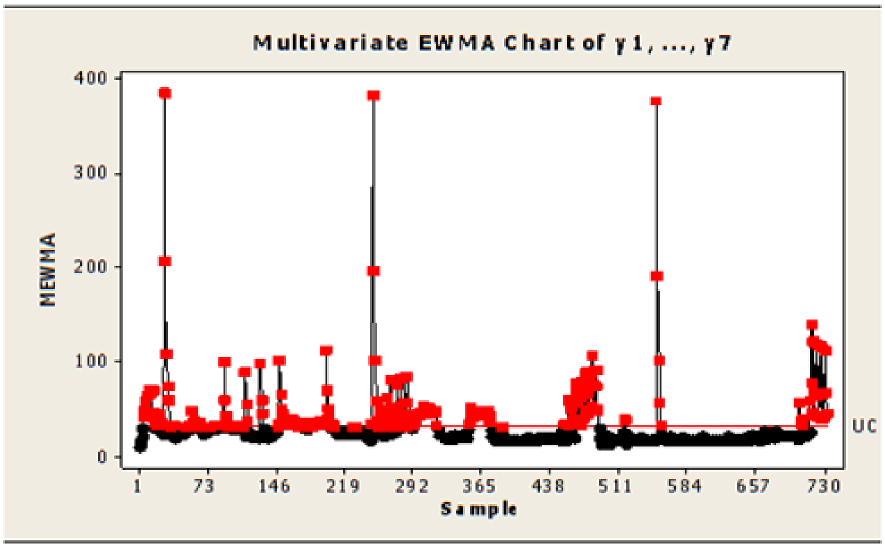

Figure2.MEWMAchart of y1, …, y7 and t3.1, …, t4.5.

The MEWMA chart of y1, …, y7 and t3.1, …, t4.5 can be summarized as follows:

- The upper control limit is 30.9. Therefore, we expect the MEWMA statistics to fall below 30.9.
- Test results indicate that 256 points beyond the control limits.
- Test results indicate that the process is in- control for 476 points and out –of control for 256 points. Then the out-of-control rate 34.97% and the in-control rate 65.03%.

4.1.3 MEWMA control chart of Z1, …, Z4 and t6.1, …, t6.8

Test Results for MEWMA Chart of Z1, …, Z4 and t6.1, …, t6.8

TEST. One point beyond control limits.

Test Failed at points:

11	15	16	17	18	19	20	21	22	23	24	25	26	27	28
29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
44	47	49	51	53	55	57	58	59	60	61	62	63	64	65
66	70	72	74	75	76	77	78	79	80	81	82	83	84	85
86	87	88	89	90	92	93	94	95	96	97	98	99	101	102
103	104	105	106	107	108	109	110	111	115	119	124	127	129	132
134	135	139	140	141	142	143	144	148	152	157	160	162	165	167
168	172	173	174	175	176	177	181	185	190	193	195	198	200	201
208	213	219	222	226	227	228	229	230	231	232	234	235	237	238
239	240	241	242	243	244	245	246	248	249	250	251	252	253	254
255	256	258	259	260	261	262	263	265	266	269	270	272	273	274
276	277	278	280	281	282	284	285	286	303	310	336	349	353	354
355	374	385	386	387	414	471	475	476	487	488	491	493	495	496
497	498	499	500	501	502	503	504	505	506	507	508	509	510	511
512	513	514	515	516	517	518	519	520	521	523	525	527	529	530
531	532	533	534	535	536	537	538	539	540	541	542	543	544	545
546	547	548	549	550	551	552	556	557	558	559	560	561	562	563
564	565	566	567	568	569	570	571	572	573	574	575	577	578	579
581	582	583	584	585	586	587	588	589	590	591	592	593	594	595
596	597	598	599	600	601	602	603	604	605	606	607	608	609	610
611	612	613	614	615	616	617	618	619	620	621	622	623	624	625
626	627	628	629	630	631	632	633	634	635	636	647	651	652	657
658	659	660	663	664	665	666	667	668	669	670	671	672	673	674
675	677	678	679	680	681	682	683	684	685	686	687	688	689	690
691	692	693	694	695	696	697	698	699	700	703	704	705	706	707

708	709	710	711	712	713	714	715	716	717	718	719	720	721	722
723	724	725	726	727	728	729	730	731	732					

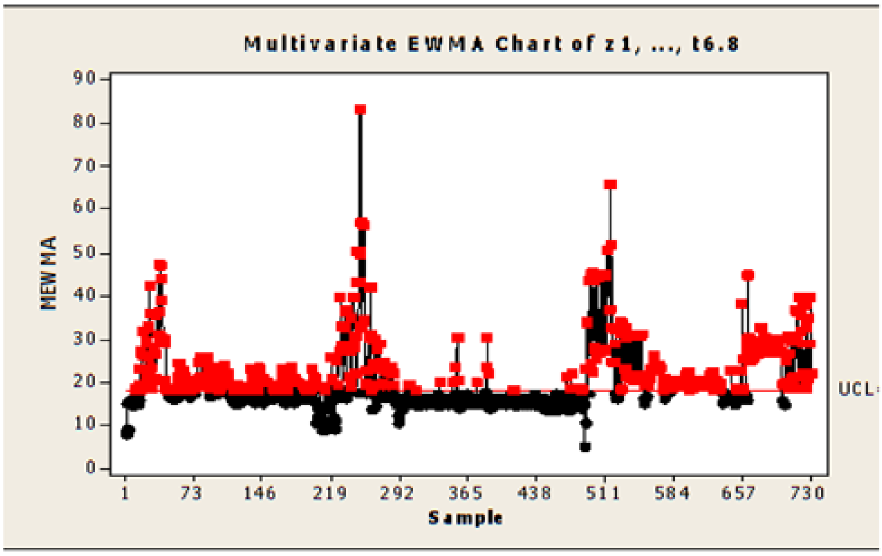

Figure3.MEWMA chart of Z1, …, Z4 and t6.1, …, t6.8.

The MEWMA chart of Z1, …, Z4 and t6.1, …, t6.8 can be summarized as follows:

- The upper control limit is 18.05 Therefore, we expect the MEWMA statistics to fall below 18.05.
- Test results indicate 401 points through beyond the control limits.
- Test results indicate that the process is in- control for 331 points and out –of control for 401 points. Then the out-of- control rate 54.78% and the in-control rate 45.22%.

4.2 Hotelling T^2 chart

A Hotelling T^2 chart consists of:

- Plotted points, each of which represents T^2 statistic for each observation.
- A center line (green), which is the median of the theoretical distribution of T^2 statistic.
- Control limits (red), which provide a visual means for assessing whether the process is in-control. The control limits represent the expected variation.

MINITAB marks points outside of the control limits with a red symbol.

MINITAB indicates which points is out-of-control by using decomposition of T^2 statistic, along with the *P*-value for each significant variable.

4.2.1 T squared chart of X1, …, X16 and t1.1, …, t2.5

Test results for T squared chart of X1, …, X16 and t1.1, …, t2.5

TEST. One point beyond control limits.

Test Failed at points: (Less Than LCL)

114	118	123	128	133	134	138	140	143	147
151	156	161	166	167	171	173	349	352	353
354	361	373	374	378	384	385	386	576	578
582	586	590	594	598	600	604	608	612	616
620	624	626	630	634	638	642	646	650	652
656	660								

Test Failed at points: (Greater Than UCL)

13	20	30	40	43	45	48	50	52	56
60	66	80	100	245	247	250	252	254	259
261	265	276	448	458	483	636	647	657	658

663	664	703	714	715	718	721	724	727	730

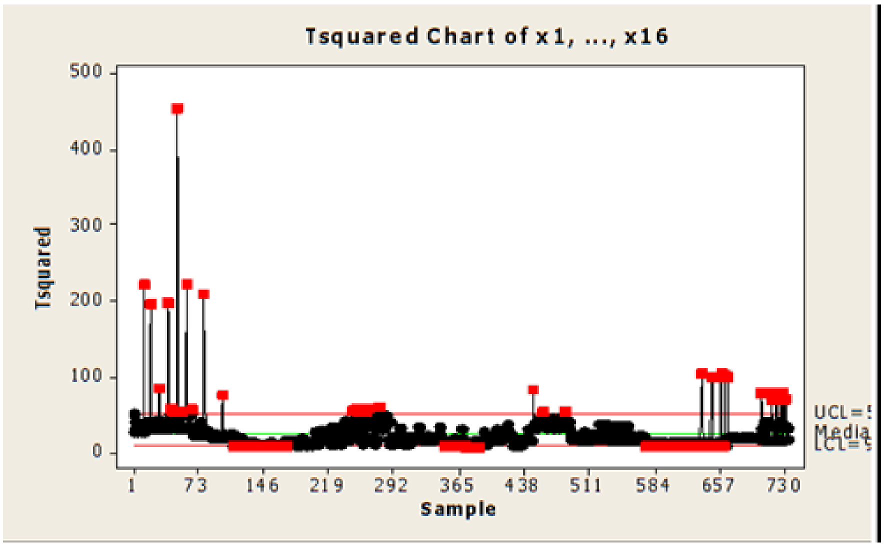

Figure 4. T^2 chart of X1, …, X16 and t1.1, …, t2.5.

The HotellingT^2 chart of X1, …, X16 and t1.1, …, t2.5 can be summarized as follows:

- The lower and upper control limits are 9.7 and 52, respectively. Therefore, we expect the T^2 Statistics to fall between 9.7 and 52. The center line or median, is 25.3.
- Test results indicate that 52 points less than LCL, for example, point 114 exceeds the lower control limit.
- Test results indicate that 40 points greater than UCL, for example, the test results indicate that point13 exceeds the upper control limit.
- Test results indicate 92 Point through beyond the control limits. Then the out-of-control rate 12.6% and the in-control rate 87.4%.

4.2.2 T squared chart of y1, …, y7 and t3.1, …, t4.5

Test results for T squared chart of y1, …, y7 and t3.1, …, t4.5

TEST. One point beyond control limits.

Test Failed at points: (Greater Than UCL)

28	91	114	130	150	200	250	256	264	268
551	703	714	715	718	721	724	727	730	

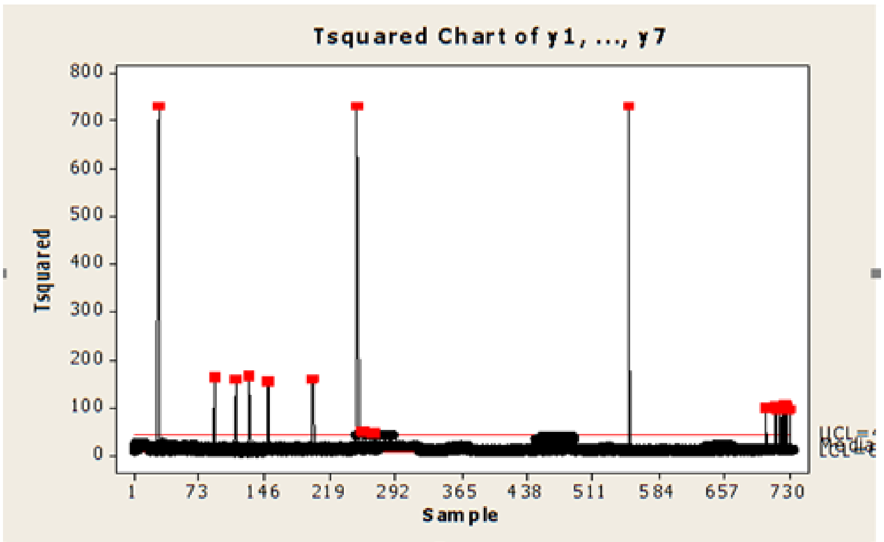

Figure 5. T^2 chart of y1, …, y7 and t3.1, …, t4.5.

The HotellingT^2 chart of y1, …, y7 and t3.1, …, t4.5 can be summarized as follows:

- The lower and upper control limits are 0.2 and 43.6, respectively. Therefore, we expect the T^2statistics to fall between 0.2 and 43.6. The center line, or median, is 19.3.
- Test results indicate that 19 Point greater than UCL, for example test results indicate that Point 91 exceeds the upper control limit.
- Test results indicate 19 Point that are beyond the control limit. Then the out of control rate 2.59% and the in-control rate 97.41%.

4.2.3 T squared chart of Z1, …, Z4 and t6.1, …, t6.8

Test results for T squared chart of Z1, …, Z4 and t6.1, …, t6.8

TEST. One point beyond control limits.

Test Failed at points: (Greater Than UCL)

245	248	250	252	254	259	261	265	269	272	276
280	284	489	491	493	495	497	515	517	519	

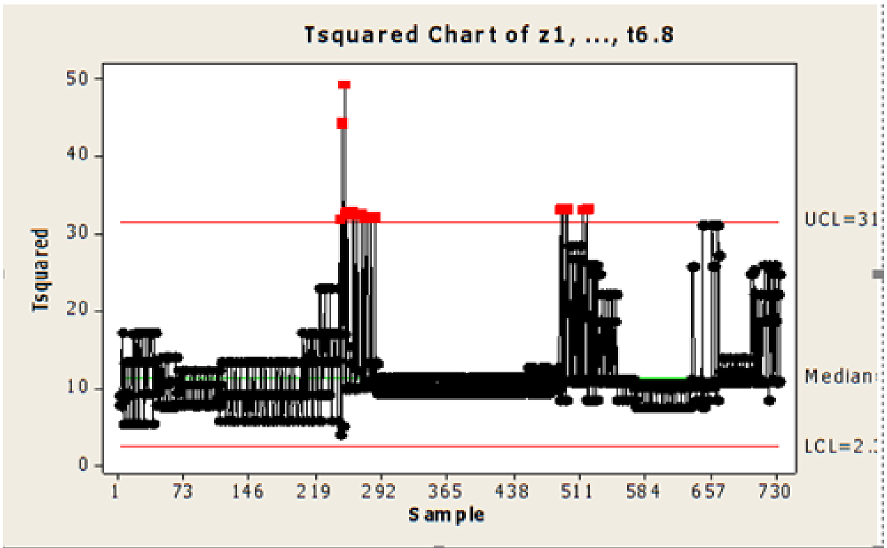

Figure 6. T^2 chart of Z1, …, Z4 and t6.1, …, t6.8

The HotellingT^2 chart of Z1, …, Z4 and t6.1, …, t6.8 can be Summarized as follows:

- The lower and upper control limits are 2.37 and 31.63, respectively. Therefore, we expect the T^2statistics to fall between 2.37 and 31.63. The center line, or median, is 11.35.
- Test results indicate that 21 Point greater than UCL, for example test results indicate that Point 245 exceeds the upper control limit.
- Test results indicate 21 Point that are beyond the control limit. Then the out of control rate 2.87% and the in-control rate 97.13%.

V. COMPARISON RESULTS OF APPLICATION

The application is shown that in High process stage, test results of MEWMA chart indicate that the out-of-control percentage 67.08% and the in-control percentage 32.92%, and it shown that in Low process stage, test results of MEWMA chart indicate that the out-of-control percentage 34.97% and the in-control percentage 65.03%. It is shown that in the Evaporation and Prilling stage, test results of MEWMA chart indicates that the out-of-control percentage 54.78% and the in-control percentage 45.22%.While the application is shown that in HotellingT^2 chart, in the High process stage, test results indicate that the out-of-control percentage 87.4%and the in-control percentage 12.6%, while in the Low process stage, the out-of-control percentage2.59% and the in-control percentage97.41% and in the Evaporation and Prilling stage, the out-of-control percentage 2.87% and the in-control percentage 97.13%.

VI. CONCLUSIONS

The results allow us to determine whether the joint process variability is in-control or out-of-control. It is shown that the out-of-control and in-control percentage changes by using difference values of *r*. It was shown that there is a relationship between the value of r and the out-of-control percentage, the out-of-control percentage increased by increasing the value of *r*and ARL. The results are shown that in the design of MEWMA control charts, small values of *r* are more efficient in detecting small process mean shifts and large values of *r* are more efficient in detecting large process mean shifts. Hotelling's T^2 charts used to determine whether or not the process mean vector (A vector of the process means that accounts for the mean of each charted variable) for two or more variables is in-control. An in-control process exhibits only random variation with the control limits. An out-of-control process exhibits unusually variation, which may be due to the process of assignable causes (unusual occurrences that are not normally part of the process).T^2 charts allow us to simultaneously monitor whether two or more related variables are in control.

Finally,

- On using the MEWMA chart to determine whether or not the process in control, the company should choose small values of r to detect small process mean shifts and choose large values of r to detect large process mean shifts.
- The company should use multivariate Hotelling's T^2 quality control chart to monitor the quality of the urea production. Too, the company should use the Hotelling's T^2 chart to determine variables which causes the out-of-control signals.

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