On A Subclass of Harmonic Univalent Functions Defined By Generalized Derivative Operator

N. D. Sangle
Department of Mathematics, Annasaheb Dange College of Engineering, Ashta, Sangali, (M.S) India 416301.

Y. P. Yadav
Department of Mathematics, S. B. Patil Polytechnic, Indapur, Pune, (M.S) India 413106

ABSTRACT
In the present paper, a subclass of harmonic univalent functions is defined using generalized derivative operator and we have obtained among others results like, coefficient inequalities, distortion theorem and convex combination.

2000 AMS subject classification: Primary 30C45, Secondary 30C50.

Keywords: Univalent functions, Harmonic functions, Derivative operator, Convex Combinations, Distortion Bounds

1. INTRODUCTION

A continuous function $f(z)$ is said to be a complex-valued harmonic function in a simply connected domain D in complex plane C if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are real harmonic in D. Such functions can be expressed as

$$f(z)=h(z)+\overline{g(z)} \qquad (1.1)$$

where $h(z)$ and $g(z)$ are analytic in D. We call $h(z)$ as analytic part and $g(z)$ as co-analytic part of $f(z)$. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in D is that $|h^{'}(z)|>|g^{'}(z)|$ for all z in D. [2]

Let S_H be the family of functions of the form (1.1) that are harmonic, univalent and orientation preserving in the open unit disk $U=\{z:|z|<1\}$, so that $f(z)=h(z)+\overline{g(z)}$ is normalized by $f(0)=h(0)=f_z(0)-1=0$. Further $f(z)=h(z)+\overline{g(z)}$ can be uniquely determined by the coefficients of power series expansions.

$$h(z)=z+\sum_{p=2}^{\infty}a_p z^p, \qquad g(z)=\sum_{p=1}^{\infty}b_p z^p, \quad z\in U, \quad |b_1|<1, \qquad (1.2)$$

where $a_p\in C$ for $p=2,3,4,...$ and $b_p\in C$ for $p=1,2,3,...$

We note that this family S_H was investigated and studied by Clunie and Sheil-Small [2] and it reduces to the well-known family S the class of all normalized analytic univalent functions $h(z)$ given in (1.2), whenever the co-analytic part $g(z)$ of $f(z)$ is identically zero.

Let $\overline{S_H}$ denote the subfamily of S_H consisting of harmonic functions of the form

$$f_n(z)=h(z)+\overline{g_n(z)}$$

Where $$\mathrm{h(z)}=z+\sum_{p=2}^{\infty}a_p z^p, \quad g_n(z)=(-1)^n\sum_{p=1}^{\infty}b_p z^p, \quad z\in U, \quad |b_1|<1. \qquad (1.3)$$

For $f(z)=h(z)+\overline{g(z)}$ given by(1.1), we define the derivative operator introduced by Shaqsi and Darus [8] of $f(z)$ as,

$$D^n_{m,\lambda} f(z) = D^n_{m,\lambda} h(z) + (-1)^n \overline{D^n_{m,\lambda} g(z)} , \qquad (1.4)$$

where

$$D^n_{m,\lambda} h(z) = z + \sum_{p=2}^{\infty} \left[1+(p-1)\lambda\right]^n C(m,p) a_p z^p$$

$$D^n_{m,\lambda} g(z) = \sum_{p=1}^{\infty} \left[1+(p-1)\lambda\right]^n C(m,p) b_p z^p, \ |b_1|<1, \ C(m,p) = C\begin{pmatrix} p+m-1 \\ m \end{pmatrix}.$$

Definition: The function $f(z) = h(z) + \overline{g(z)}$ defined by (1.2) is in the class $S_H(n,m,k,\lambda,\beta)$ if

$$\operatorname{Re}\left\{\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)}\right\} \geq k\left|\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)} - 1\right| + \beta \qquad (1.5)$$

where $0 \leq k < \infty, \ 0 \leq \beta < 1.$

Also let

$$\overline{S_H}(n,m,k,\lambda,\beta) = S_H(n,m,k,\lambda,\beta) \cap \overline{S_H} \qquad (1.6)$$

We note that by specializing the parameter, especially when $k = 0, \ S_H(n,m,k,\lambda,\beta)$ reduces to well-known family of starlike harmonic functions of order β . In recent years many researchers have studied various subclasses of S_H for example [1],[3],[4],[6]and [8].

In the present paper we aim at systematic study of basic properties, in particular coefficient bound , distortion theorem and extreme points of aforementioned subclass of harmonic functions.

2. MAIN RESULTS

Theorem1: Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.2). If condition

$$\sum_{p=1}^{\infty}\left\{\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)|a_p| + \frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)|b_p|\right\} \leq 2 \quad \text{where} \qquad (2.1)$$

$a_1 = 1, \ 0 \leq \beta < 1, \ \ 0 \leq k < \infty, \ \ n \in N \cup \{0\},$ then $f(z)$ is sense-preserving harmonic univalent in U and $f \in S_H(n,m,k,\lambda,\beta).$

Proof: If the inequality (2.1) holds for coefficients of $f(z) = h(z) + \overline{g(z)}$ then by (1.2), $f(z)$ is orientation preserving and harmonic univalent in U. Now it remains to show that $f \in S_H(n,m,k,\lambda,\beta)$. According to (1.4) and (1.5) we have

$$\operatorname{Re}\left\{\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)}\right\} \geq k\left|\frac{D^{n+1}_{m,\lambda} f(z)}{D^n_{m,\lambda} f(z)} - 1\right| + \beta$$

which is equivalent to $\operatorname{Re}\left(\frac{A(z)}{B(z)}\right) > \beta$

where $A(z)=\left(1+k\right)D_{m,\lambda}^{n+1}f(z)-kD_{m,\lambda}^{n}f(z)$ and $B(z)=D_{m,\lambda}^{n}f(z)$

Using the fact that, $\mathrm{Re}(w)>\beta$ if $\left|1-\beta+w\right|\ge\left|1+\beta-w\right|$ it suffices to show that

$\left|A(z)+\left(1-\beta\right)B(z)\right|\ge\left|A(z)-\left(1+\beta\right)B(z)\right|$ substituting values of A(z) and B(z) with simple calculations we led to

$$=\left|(2-\beta)z+\sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k+1-\beta\right]C(m,p)a_{p}z^{p}-(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-1+\beta\right]C(m,p)\bar{b}_{p}\bar{z}^{p}\right|$$

$$-\left|\beta z+\sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k+1-\beta\right]C(m,p)a_{p}z^{p}+(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-1+\beta\right]C(m,p)\bar{b}_{p}\bar{z}^{p}\right|$$

$$\ge 2\left(1-\beta\right)|z|-\sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[2\left(1+k\right)\left[1+(p-1)\lambda\right]-2k-2\beta\right]C(m,p)\left|a_{p}\right||z|^{p}$$

$$-(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[2\left(1+k\right)\left[1+(p-1)\lambda\right]+2k+2\beta\right]C(m,p)\left|\bar{b}_{p}\right|\left|\bar{z}\right|^{p}$$

$$\ge 2\left(1-\beta\right)|z|\left\{1-\sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\frac{\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)\left|a_{p}\right||z|^{p-1}\right.$$

$$\left.-(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\frac{\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)\left|\bar{b}_{p}\right|\left|\bar{z}\right|^{p-1}\right\}\ge 0.$$

By assumption. Hence proof is completed.

The functions

$$f(z)=z+\sum_{p=2}^{\infty}\left[\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}\right]x_{p}z^{p}+\sum_{p=1}^{\infty}\left[\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}\right]\bar{y}_{p}\bar{z}^{p}$$

where $$\sum_{p=2}^{\infty}\left|x_{p}\right|+\sum_{p=1}^{\infty}\left|y_{p}\right|=1 \qquad (2.3)$$

shows that the coefficient bound given (2.1) is sharp.

Theorem 2: Let $f_{n}(z)=h(z)+\overline{g_{n}(z)}$ be so that $h(z)$ and $g_{n}(z)$ given by (1.6). Then $f_{n}\in\overline{S_{H}}(n,m,k,\lambda,\beta)$ if and only if

$$\sum_{p=1}^{\infty}\left\{\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)\left|a_{p}\right|+\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)\left|\bar{b}_{p}\right|\right\}\le 2, \qquad (2.4)$$

where $a_{1}=1, 0\le\beta<1, 0\le k<\infty.$

Proof: The if part follows form Theorem1 with the fact the $\overline{S_H}(n,m,k,\lambda,\beta) \subset S_H(n,m,k,\lambda,\beta)$. For only if part, we show that $f_n \notin \overline{S_H}(n,m,k,\lambda,\beta)$ if the condition (2.4) is not satisfied. Note that necessary and sufficient condition for Let $f_n = h + \overline{g_n}$ given by (1.6) to be in $\overline{S_H}(n,m,k,\lambda,\beta)$ is that

$$\operatorname{Re}\left\{\frac{D_{m,\lambda}^{n+1} f(z)}{D_{m,\lambda}^{n} f(z)}\right\} \geq k\left|\frac{D_{m,\lambda}^{n+1} f(z)}{D_{m,\lambda}^{n} f(z)} - 1\right| + \beta$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{(1+k)D_{m,\lambda}^{n+1} f(z) + (k-\beta)D_{m,\lambda}^{n} f(z)}{D_{m,\lambda}^{n} f(z)}\right\}$$

$$= \operatorname{Re}\left\{\frac{\begin{array}{c}(1-\beta)z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]C(m,p)a_p z^p \\ -(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]C(m,p)\bar{b}_p \bar{z}^p\end{array}}{\begin{array}{c}z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)a_p z^p \\ +(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)\bar{b}_p \bar{z}^p\end{array}}\right\} > 0.$$

The above conditions must hold for all values of z , $|z| = r < 1$. Choosing z on positive axis where $0 \leq |z| = r < 1$. we have

$$\frac{\begin{array}{c}(1-\beta)z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]C(m,p)a_p r^{p-1} \\ -(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]C(m,p)\bar{b}_p \overline{r^{p-1}}\end{array}}{z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)a_p r^{p-1} + (-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)\bar{b}_p \overline{r^{p-1}}} \geq 0. \quad (2.5)$$

or equivalently if the condition (2.4) dose not hold then the numerator in (2.5) is negative for r sufficiently close to 1.

Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative .This contradicts that required condition for $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ and hence proof is completed.

Theorem 3: Let f_n be given by (1.6). Then $f_n \in \overline{S_H}(k,\beta;n)$ if and only if

$$f_n(z) = \sum_{p=1}^{\infty}\left(x_p h_p(z) + y_p g_{n_p}(z)\right)$$

where, $h_1(z)=1,\ \ h_p(z)=z-\dfrac{(1-\beta)}{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}z^p,\quad p=2,3,4,\ldots$

$$g_{n_p}(z)=z+(-1)^{n-1}\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}\overline{z}^p,\qquad p=1,2,3,\ldots \text{ and}$$

$x_p\geq 0, y_p\geq 0$, $x_1=1-\sum_{p=2}^{\infty}\left(x_p+y_p\right)\geq 0$.

In particular, the extreme points of $\overline{S_H}(n,m,k,\lambda,\beta)$ are $\{h_n\}$ and $\{g_{n_p}\}$.

Proof: Let

$$f_n(z)=\sum_{p=1}^{\infty}\left(x_ph_p(z)+y_pg_{n_p}(z)\right)$$

$$=\sum_{p=2}^{\infty}\left(x_p+y_p\right)-\sum_{p=2}^{\infty}\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}x_pz^p$$

$$+(-1)^{n-1}\sum_{p=1}^{\infty}\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}y_p\overline{z}^p$$

Then

$$=\sum_{p=2}^{\infty}\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}\left|a_p\right|$$

$$+\sum_{p=1}^{\infty}\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}\left|b_p\right|$$

$$=\sum_{p=2}^{\infty}x_p+\sum_{p=1}^{\infty}y_p=1-x_1\leq 1$$

and so $f_n\in\overline{S_H}(n,m,k,\lambda,\beta)$.

Conversely, suppose that $f_n\in\overline{S_H}(n,m,k,\lambda,\beta)$.

Setting

$$x_p = \frac{\left[1+(p-1)\lambda\right]^n \left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)} a_p, \; p = 2,3,\ldots$$

$$y_p = \frac{\left[1+(p-1)\lambda\right]^n \left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)} b_p, \; p = 1,2,3,\ldots$$

where $\sum_{p=1}^{\infty}\left(x_p + y_p\right) = 1$ we obtain $f_n(z) = \sum_{p=1}^{\infty}\left(x_p h_p(z) + y_p g_{n_p}(z)\right)$ as required.

Theorem 4: Let $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ then for $|z| = r < 1$

we have

$$|f_n(z)| \le (1+|b_1|)r + \frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta} - \frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\} r^2$$

and

$$|f_n(z)| \ge (1-|b_1|)r - \frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta} - \frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\} r^2$$

Proof. *Let* $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ .Taking absolute value of f_n we obtain

$$|f_n(z)| \le (1+|b_1|)r + \sum_{p=2}^{\infty}\left(|a_p| + |b_p|\right) r^p$$

$$\le (1+|b_1|)r + \sum_{p=2}^{\infty}\left(|a_p| + |b_p|\right) r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\left\{\sum_{p=2}^{\infty}\frac{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}{(1-\beta)}\left(|a_p| + |b_p|\right)\right\} r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\sum_{p=2}^{\infty}\left(\frac{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}{(1-\beta)}|a_p| + \frac{(2\lambda)^n\left[(1+k)(1+\lambda)+k+\beta\right]}{(1-\beta)}|b_p|\right) r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\sum_{p=2}^{\infty}\left(1 - \frac{(2\lambda)^n\left[(1+k)(1+\lambda)+k+\beta\right]}{(1-\beta)}|b_p|\right) r^2$$

$$\le (1+|b_1|)r + \frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta} - \frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\} r^2.$$

The forthcoming result follows from left hand inequality in Theorem 2.4.

Theorem 5: The class of $\overline{S_H}(n,m,k,\lambda,\beta)$ is closed under convex combination.

Proof: For $i=1,2,3,\ldots$ suppose $f_{n_i}(z)\in\overline{S_H}(n,m,k,\lambda,\beta)$ where

$$f_{n_i}=z-\sum_{p=2}^{\infty}|a_{ip}|z^p+(-1)^n\sum_{p=1}^{\infty}|b_{ip}|\bar{z}^p$$

then by Theorem 2

$$\sum_{p=2}^{\infty}\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]-k-\beta]}{(1-\beta)}|a_{ip}|$$
$$+\sum_{p=2}^{\infty}\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]+k+\beta]}{(1-\beta)}|b_{ip}|\le 1. \qquad (2.6)$$

For $\sum_{i=1}^{\infty}t_i=1,\ 0\le t_i\le 1,$ the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{n_i}(z)=z-\sum_{p=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{ip}|\right)z^p+(-1)^n\sum_{p=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{ip}|\right)\bar{z}^p$$

hence by (2.6)

$$\sum_{p=2}^{\infty}\left(\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]-k-\beta]}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i|a_{ip}|\right)$$
$$+\sum_{p=1}^{\infty}\left(\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]+k+\beta]}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i|b_{ip}|\right)$$

$$=\sum_{i=i}^{\infty}t_i\left(\sum_{p=2}^{\infty}\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]-k-\beta]}{(1-\beta)}|a_{ip}|+\sum_{p=1}^{\infty}\frac{[1+(p-1)\lambda]^n[(1+k)[1+(p-1)\lambda]+k+\beta]}{(1-\beta)}|a_{ip}|\right)$$

$$\le\sum_{i=i}^{\infty}t_i\le 1$$

and therefore $\sum_{i=1}^{\infty}t_if_{n_i}(z)\in\overline{S_H}(n,m,k,\lambda,\beta)$

This completes the proof.

REFERENCES

1. O. P. Ahuja, R. Aghalary and S. B. Joshi, Harmonic univalent functions associated with k- uniformly starlike functions, *Math Sci. Res.* J. **,9 (1) (**2005), 9-17.

2. J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn A. I. Math.*, **9** (1984), 3-25.

3. J. M. Jahangiri, Harmonic functions starlike in the unit disc, *J. Math. Anal. Appl.*, **235** (1999), 470-477.

4. J. M. Jahangiri, Y. C. Kim and H. M. Srivastava, Construction of a certain class of harmonic close-to convex functions associated with Alexander integral transform, *Integral Trans and Spec Funct.*, **14** (2003), 237-242.

5. J. M. Jahangiri, G. Murgusundarmoorthy and K. Vijay, Salagean –type harmonic univalent functions, South *J. Pure and Appl. Math.*, **2** (2002), 77-82.

6. Al-Oboudi, F. M. On univalent functions defined by generalized Salagean operator, *IJMMS*, **27**(2004), 1429-1436.

7. G. S. Salagean, Subclass of univalent functions, *Lect Notes in Math. Springer-Verlag*, **1013** (1983), 362-372.

8. K. Al-Shaqsi and M. Darus, Differential Sandwich theorems with generalized derivative oprator, *Inter . J. Comp. Math. Sci.*, **2,** (2) (2008), 75-78.

9. H. Silverman, Harmonic univalent functions with negative coefficients, J. *Math Anal Appl.*, **220**(1998), 283-289.

10. S. Yalcin, On certain harmonic univalent functions defined by Salagean derivative, *Soochow J.Math.*, **31 (3) (**2005), 321-331.